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Sveska sa vježbi iz
Matematike II (I dio)

Odsjeci: Inžinjerski dizajn proizvoda, Inžinjerska ekologija, Menadžment proizvodnim tehnologijama, Održavanje

Zbirke zadataka za dodatno usavršavanje i napredovanje:

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Osnovne formule iz Matematike II

Dio tablica integrala.

1. $\int u^a \mathrm{d}u = \frac{u^{a+1}}{a+1} + C,\ a \neq -1.$
2. $\int u^{-1}\mathrm{d}u = \int \frac{\mathrm{d}u}{u} = \int \frac{u'}{u}\mathrm{d}x = \ln|u| + C.$
3. $\int a^u \mathrm{d}u = \frac{a^u}{\ln|a|} + C;\ \int e^u \mathrm{d}u = e^u + C.$
4. $\int \sin \mathrm{d}u = -\cos u + C.$
5. $\int \cos \mathrm{d}u = \sin u + C.$
6. $\int \frac{1}{\cos^2 u}\mathrm{d}u = \mathrm{tg}\, u + C.$
7. $\int \frac{1}{\sin^2 u}\mathrm{d}u = -\mathrm{ctg}\, u + C.$
8. $\int \frac{\mathrm{d}u}{u^2+a^2} = \frac{1}{a}\mathrm{arc\,tg}\,\frac{u}{a} + C.$
9. $\int \frac{\mathrm{d}u}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C.$
10. $\int \frac{\mathrm{d}u}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C.$
11. $\int \frac{\mathrm{d}u}{\sqrt{u^2+a}} = \ln|u+\sqrt{u^2+a}| + C.$

Newton-Leibnizova formula.

$$\int_a^b f(u)\mathrm{d}u = \int f(u)\mathrm{d}u\Big|_a^b = F(u)|_a^b = F(b)-F(a), \text{ gdje je } F'(u)=f(u).$$

Osobine određenih integrala.

1. $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x.$
2. $\int_a^a f(x)\mathrm{d}x = 0.$
3. $\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x.$
4. $\int_a^b [f_1(x)+f_2(x)-f_3(x)]\mathrm{d}x = \int_a^b f_1(x)\mathrm{d}x + \int_a^b f_2(x)\mathrm{d}x - \int_a^b f_3(x)\mathrm{d}x.$
5. $\int_a^b c\, f(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x.$

Smjena promjenjivih u određenom integralu.

$$\int_a^b f(x)\mathrm{d}x = \left|\begin{array}{ll} x=\varphi(t) & x=a \Rightarrow a=\varphi(\alpha)\Rightarrow t=\alpha \\ dx=\varphi'(t)\mathrm{d}t & x=b\Rightarrow b=\varphi(\beta)\Rightarrow t=\beta\end{array}\right| = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\mathrm{d}t = \int_\alpha^\beta h(t)\mathrm{d}t$$

Nepravi integrali. $\int\limits_a^{+\infty} f(x)\mathrm{d}x = \lim\limits_{b\to\infty}\int_a^b f(x)\mathrm{d}x,\ \int\limits_{-\infty}^{b} f(x)\mathrm{d}x = \lim\limits_{a\to-\infty}\int_a^b f(x)\mathrm{d}x$,...

Računanje površine ravne figure. U zavisnosti od izgleda slike: $P=\int_a^b f(x)\mathrm{d}x,\ P=\int_c^d g(y)\mathrm{d}y,$ $P=-\int_a^b f(x)\mathrm{d}x,\ P=\int_a^b[\eta(x)-\mu(x)]\mathrm{d}x,\ P=\int_c^d[g(y)-h(y)]\mathrm{d}y,$ ...

Zapremina rotacionog tijela. Ako, kriva data u parametarskom obliku $C:\left\{\begin{array}{c} x=\eta(t)\\ y=\mu(t)\\ t_1\le t\le t_2\end{array}\right.$ rotira oko x-ose, zapremine se računa po formuli

$$V_x=\pi\int_{t_1}^{t_2}[\mu(t)]^2|\eta'(t)|\mathrm{d}t.$$

Ista kriva ako rotira oko y-ose, $V_y=\pi\int_{t_1}^{t_2}[\eta(t)]^2|\mu'(t)|\mathrm{d}t$. Iz ove dvije formule, za funkcije $y=f(x)$ i $x=g(y)$, slijedi $V_x=\pi\int_a^b[f(x)]^2\mathrm{d}x$ i $V_y=\pi\int_c^d[g(y)]^2\mathrm{d}y$.

Dužina luka krive. $C:\left\{\begin{array}{c} x=\eta(t)\\ y=\mu(t)\\ t_1\le t\le t_2\end{array}\right.,\ \ell=\int\limits_{t_1}^{t_2}\sqrt{[\eta'(t)]^2+[\mu'(t)]^2}\mathrm{d}t;$

$C:\left\{\begin{array}{c} y=f(x)\\ a\le x\le b\end{array}\right.,\ \ell=\int\limits_a^b\sqrt{1+[y'(x)]^2}\mathrm{d}x;\quad C:\left\{\begin{array}{c} x=g(y)\\ c\le y\le d\end{array}\right.,\ \ell=\int\limits_c^d\sqrt{1+[g'(y)]^2}\mathrm{d}y;$

Komplanacija obrtne površi. Površina omotača tijela dobijenog rotacijom krive $C:\left\{\begin{array}{c} x=\eta(t)\\ y=\mu(t)\\ t_1\le t\le t_2\end{array}\right.$, oko x-ose, se računa po formuli: $P=2\pi\int\limits_{t_1}^{t_2}|\mu(t)|\sqrt{[\eta'(t)]^2+[\mu'(t)]^2}\mathrm{d}t;$

$C:\left\{\begin{array}{c} y=f(x)\\ a\le x\le b\end{array}\right.,\ P=2\pi\int\limits_a^b|f(x)|\sqrt{1+[f'(x)]^2}\mathrm{d}x;$ ...

Funkcije dvije nazavisno promjenjive. ...

Parcijalni izvodi f-ja više pomjenjivih. $z=f(x,y),\ z'_x=\frac{\partial z}{\partial x}=\lim\limits_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$ ...

Diferenciranje funkcija više promjenjivih. $u=f(x,y,z),\ du=\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y+\frac{\partial u}{\partial z}\mathrm{d}z$ ...

Diferenciranje složenih funkcija. ...

Parcijalni izvodi višeg reda složenih funkcija. ...

Ekstremne vrijednosti f-ja dvije promjenjive. ...

Uslovni ekstremi f-ja dvije promjenjive. ...

Jednačina tangentne ravni i jednačina normale na površ. Ako je S u obliku $F(x,y,z)=0$

$$\alpha:\ F'_x(x_0,y_0,z_0)(x-x_0)+F'_y(x_0,y_0,z_0)(y-y_0)+F'_z(x_0,y_0,z_0)(z-z_0)=0$$

$$n:\ \frac{x-x_0}{F'_x(x_0,y_0,z_0)}=\frac{y-y_0}{F'_y(x_0,y_0,z_0)}=\frac{z-z_0}{F'_z(x_0,y_0,z_0)}$$

Dvojni integrali.

$$\iint\limits_D f(x,y)=\int\limits_a^b\mathrm{d}x\int\limits_{g(x)}^{h(x)}f(x,y)\mathrm{d}y=\int\limits_a^b\left[\int\limits_{g(x)}^{h(x)}f(x,y)\mathrm{d}y\right]\mathrm{d}x,$$

$$\iint\limits_D f(x,y)=\int\limits_c^d\mathrm{d}y\int\limits_{\eta(y)}^{\mu(y)}f(x,y)\mathrm{d}x=\int\limits_c^d\left[\int\limits_{\eta(y)}^{\mu(y)}f(x,y)\mathrm{d}x\right]\mathrm{d}y\ ...$$

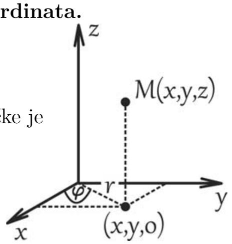
Smjena promjenjivih u dvojnim integralima. Za prelazak sa pravougaonih na polarne koordinate koristimo smjene $\left\{\begin{array}{c} x=r\cos(\varphi)\\ y=r\sin(\varphi)\\ \mathrm{d}x\mathrm{d}y=r\mathrm{d}r\mathrm{d}\varphi\end{array}\right.$, poopštene plarne koordinate su oblika $\left\{\begin{array}{c} x=a\,r\cos(\varphi),\ (a>0)\\ y=b\,r\sin(\varphi),\ (b>0)\\ \mathrm{d}x\mathrm{d}y=a\,b\,r\mathrm{d}r\mathrm{d}\varphi\end{array}\right.$, a za proizvoljne smjene $\left\{\begin{array}{c} x=\eta(u,v)\\ y=\mu(u,v)\\ \mathrm{d}x\mathrm{d}y=|J|\mathrm{d}u\mathrm{d}v\end{array}\right.$, gdje je J Jakobijan,

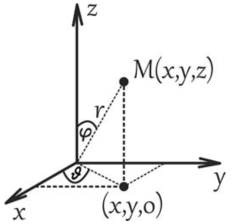
$$J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$$

Trojni integrali. ...

Računanje trojnih integrala uvođenjem cilindričnih i sfernih koordinata.

Na cilindrične koordinate prelazimo pomoću $\left\{\begin{array}{c} x = r\cos(\varphi) \\ y = r\sin(\varphi) \\ z = z \\ \mathrm{d}x\mathrm{d}y\mathrm{d}z = r\mathrm{d}r\mathrm{d}\varphi\mathrm{d}z \end{array}\right.$, opis tačke je





Za prelazak sa pravougaonih na sferne koordinate koristimo sljedeće smjene $\left\{\begin{array}{c} x = r\sin(\varphi)\cos(\theta) \\ y = r\sin(\varphi)\sin(\theta) \\ z = r\cos(\varphi) \\ \mathrm{d}x\mathrm{d}y\mathrm{d}z = r^2\sin(\varphi)\mathrm{d}r\mathrm{d}\varphi\mathrm{d}\theta \end{array}\right.$, (opis tačke je prikazan na slici lijevo).

Primjena dvostrukih integrala.

$$(a)\ P = \iint_D \mathrm{d}x\mathrm{d}y. \qquad (b)\ V = \iint_D f(x,y)\mathrm{d}x\mathrm{d}y.$$

Primjena trostrukih integrala.

$(a)\ V = \iiint_\Omega \mathrm{d}x\mathrm{d}y\mathrm{d}z.$

$(b)\ T(x_T, y_T, z_T),\ \ x_T = \dfrac{1}{V}\iiint_\Omega x\mathrm{d}x\mathrm{d}y\mathrm{d}z, y_T = \dfrac{1}{V}\iiint_\Omega y\mathrm{d}x\mathrm{d}y\mathrm{d}z, z_T = \dfrac{1}{V}\iiint_\Omega z\mathrm{d}x\mathrm{d}y\mathrm{d}z.$

Krivoliniski integral prve vrste (po luku).

$$C: \left\{\begin{array}{c} x = \eta(t) \\ y = \mu(t) \\ t_1 \le t \le t_2 \end{array}\right., \quad \int_C f(x,y)\mathrm{d}s = \int_{t_1}^{t_2} f(\eta(t),\mu(t))\sqrt{(\eta'(t))^2 + (\mu'(t))^2}\mathrm{d}t.$$

$$C: \left\{\begin{array}{c} y = f(x) \\ a \le x \le b \end{array}\right., \quad \int_C z(x,y)\mathrm{d}s = \int_a^b z(x,f(x))\sqrt{1 + (f'(x))^2}\mathrm{d}x.$$

Primjena krivoliniskog integrala prve vrste - Računanje površine cilindrične površi.

$$C: \left\{\begin{array}{c} F(x,y) = 0 \\ z = 0 \end{array}\right., \quad P = \int_C z(x,y)\mathrm{d}s.$$

Krivoliniski integral druge vrste (po koordinatama).

$$C: \left\{\begin{array}{c} x = \eta(t) \\ y = \mu(t) \\ t_1 \le t \le t_2 \end{array}\right., \quad \int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \int_{t_1}^{t_2} [P(\eta(t),\mu(t))\eta'(t) + Q(\eta(t),\mu(t))\mu'(t)]\mathrm{d}t.$$

$$C: \left\{\begin{array}{c} y = f(x) \\ a \le x \le b \end{array}\right., \quad \int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \int_a^b [P(x,f(x)) + Q(x,f(x))f'(x)]\mathrm{d}x.$$

Krivoliniski integral druge vrste *ovisi o smjeru puta integracije*.

Formula Greena.

$$\int_C P(x,y)\mathrm{d}x + Q(x,y)\mathrm{d}y = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y.$$

Primjena krivoliniskog integrala druge vrste - Računanje površine ravne figure.

$$P = \frac{1}{2}\int_C x\mathrm{d}y - y\mathrm{d}x.$$

Nezavisnost krivoliniskog integrala od vrste konture. Određivanje primitivnih funkcija.

..., $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$, ..., $\mathrm{d}u(x,y) = \dfrac{\partial u}{\partial x}\mathrm{d}x + \dfrac{\partial u}{\partial y}\mathrm{d}y$, ...

Površinski integral prve vrste.

D projekcija od S: $z = \eta(x,y)$ na $x0y$ - $\displaystyle\iint_S f(x,y,z)\mathrm{d}S = \iint_D f(x,y,\eta(x,y))\sqrt{1 + \left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2}\mathrm{d}x\mathrm{d}y.$

E projekcija od S: $y = \mu(x,z)$ na $x0z$ - $\displaystyle\iint_S f(x,y,z)\mathrm{d}S = \iint_E f(x,\mu(x,z),z)\sqrt{1 + \left(\frac{\partial\mu}{\partial x}\right)^2 + \left(\frac{\partial\mu}{\partial z}\right)^2}\mathrm{d}x\mathrm{d}z.$

F projekcija od S: $x = \gamma(y,z)$ na $y0z$ - $\displaystyle\iint_S f(x,y,z)\mathrm{d}S = \iint_F f(\gamma(y,z),y,z)\sqrt{1 + \left(\frac{\partial\gamma}{\partial y}\right)^2 + \left(\frac{\partial\gamma}{\partial z}\right)^2}\mathrm{d}y\mathrm{d}z.$

Površinski integral druge vrste.

Ako je integral oblika $\displaystyle\iint_S P(x,y,z)\mathrm{d}y\mathrm{d}z + Q(x,y,z)\mathrm{d}x\mathrm{d}z + R(x,y,z)\mathrm{d}x\mathrm{d}y$ obično ga podjelimo na tri dijela $\displaystyle\iint_S P(x,y,z)\mathrm{d}y\mathrm{d}z, \iint_S Q(x,y,z)\mathrm{d}x\mathrm{d}z, \iint_S R(x,y,z)\mathrm{d}x\mathrm{d}y$. Neka je $\vec{n} = (\cos\alpha, \cos\beta, \cos\gamma)$ vektor normale na površinu S, gdje su α, β i γ uglovi koje vektor normale zaklapa sa x, y i z osom. Tada

$$I_1 = \iint_S P(x,y,z)\mathrm{d}y\mathrm{d}z = \left|\begin{array}{c} S: x = \eta(y,z), \\ \text{neka je } D \text{ projekcija od } S \text{ na } y0z \text{ ravan,} \\ \text{neka je } \alpha \text{ ugao koji vektor} \\ \text{normale na } S \text{ zaklapa sa } x\text{-osom,} \end{array}\right| = \pm\iint_D P(\eta(y,z),y,z)\mathrm{d}y\mathrm{d}z \text{ gdje}$$

vrijednost za ± zavisi od $\cos(\alpha)$ ($\cos(\alpha) > 0$ stavljamo +, za $\cos(\alpha) < 0$ stavljamo -, a za $\cos(\alpha) = 0$ imamo $I_1 = 0$). Slično za I_2 i I_3

$$I_2 = \iint_S Q(x,y,z)\mathrm{d}x\mathrm{d}z = \left|\begin{array}{c} S: y = \mu(x,z), \\ \text{neka je } E \text{ projekcija od } S \text{ na } x0z \text{ ravan,} \\ \text{neka je } \beta \text{ ugao koji vektor} \\ \text{normale na } S \text{ zaklapa sa } y\text{-osom,} \end{array}\right| = \pm\iint_E Q(x,\mu(x,z),z)\mathrm{d}x\mathrm{d}z.$$

$$I_3 = \iint_S R(x,y,z)\mathrm{d}x\mathrm{d}y = \left|\begin{array}{c} S: z = \delta(x,y), \\ \text{neka je } F \text{ projekcija od } S \text{ na } x0y \text{ ravan,} \\ \text{neka je } \gamma \text{ ugao koji vektor} \\ \text{normale na } S \text{ zaklapa sa } x\text{-osom,} \end{array}\right| = \pm\iint_F R(x,y,\delta(x,y))\mathrm{d}x\mathrm{d}y.$$

Primjena površinskog integrala prve vrste - Izračunavanje površine dijela glatke površi.

$P = \displaystyle\iint_S \mathrm{d}S = \iint_D \sqrt{1 + \left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2}\mathrm{d}x\mathrm{d}y$, gdje je D projekcija od S: $z = \eta(x,y)$ na $x0y$ ravan.

Stoksova formula.

...

Formula Gauss-Ostrogradski.

$$\iint_S P(x,y,z)\mathrm{d}y\mathrm{d}z + Q(x,y,z)\mathrm{d}x\mathrm{d}z + R(x,y,z)\mathrm{d}x\mathrm{d}y = \iiint_\Omega \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

Integrali ovisni o parametru.

$$I(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\mathrm{d}x \quad\Longrightarrow\quad I'(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f'_\alpha(x,\alpha)\mathrm{d}x + b'(\alpha)f(b(\alpha),\alpha) - a'(\alpha)f(a(\alpha),\alpha).$$

Ako granice a i b ne zavise od α tada $I'(\alpha) = \displaystyle\int_a^b f'_\alpha(x,\alpha)\mathrm{d}x$.

Vektorska teorija polja.

...

Cirkulacija i fluks vektorskog polja.

$$C = \int_c \vec{v}\,\mathrm{d}\vec{r} = \int_c v_x\mathrm{d}x + v_y\mathrm{d}y + v_z\mathrm{d}z.$$

$$\Phi = \iint_S \vec{v}\vec{n}\,\mathrm{d}S = \iint_S v_x\mathrm{d}y\mathrm{d}z + v_y\mathrm{d}x\mathrm{d}z + v_z\mathrm{d}x\mathrm{d}z.$$

Neodređeni integral

1. Primitivna f-ja i neodređeni integral. Osnovne formule integriranja.

Određivanje f-je $F(x)$ iz datog diferencijala $dF(x)=f(x)dx$ (ili iz neke date derivacije $F'(x)=f(x)$) nazivamo integriranje, a traženu f-ju $F(x)$ nazivamo primitivna f-ja f-je $f(x)$. Drugim riječima efekat suprotan diferenciranju nazivamo integriranje.

Navedimo nekoliko primjera primitivnih f-ja:

- $F(x)=\cos x$ je primitivna f-ja f-je $f(x)=\sin x$ zato što je
  $F'(x)=f(x)$ $\quad((\cos x)'=\sin x)$ ili
  $dF(x)=f(x)dx$ $\quad(d(\cos x)=\sin x\,dx)$
- $F(x)=\frac{1}{4}x^4$ je primitivna f-ja f-je $f(x)=x^3$ zato što je
  $F'(x)=f(x)$ $\quad((\frac{1}{4}x^4)'=\frac{1}{4}\cdot 4x^3=x^3)$ ili
  $dF(x)=f(x)dx$ $\quad(d(\frac{1}{4}x^4)=\frac{1}{4}d(x^4)=\frac{1}{4}\cdot 4x^3dx=x^3dx)$
- $F(x)=tgx$ je primitivna f-ja f-je $f(x)=\frac{1}{\cos^2 x}$ zato što je
  $F'(x)=f(x)$ $\quad((tgx)'=\frac{1}{\cos^2 x})$ ili
  $dF(x)=f(x)dx$ $\quad(d(tgx)=\frac{1}{\cos^2 x}dx)$.
- $F(x)=\arcsin x$ je primitivna f-ja f-je $f(x)=\frac{1}{\sqrt{1-x^2}}$ zato što je
  $dF(x)=f(x)dx$ $\quad(d(\arcsin x)=\frac{1}{\sqrt{1-x^2}}dx)$ ili
  $F'(x)=f(x)$ $\quad((\arcsin x)'=\frac{1}{\sqrt{1-x^2}})$.
- $F(x)=\ln|x|$ je primitivna f-ja f-je $f(x)=\frac{1}{x}$.

ZA VJEŽBU OBJASNITI ZAŠTO.

Svaka neprekidna f-ja f-je $f(x)$ ima beskonačno mnogo različitih primitivnih f-ja, koje se jedna od druge razlikuju u članu koji predstavlja konstantu:
ako je $F(x)$ primitivna f-ja f-je $f(x)$ (tj. ako je $F'(x)=f(x)$) tada je i $F(x)+C$ primitivna f-ja od $f(x)$, gdje je C proizvoljna konstanta. Zašto? Zato što

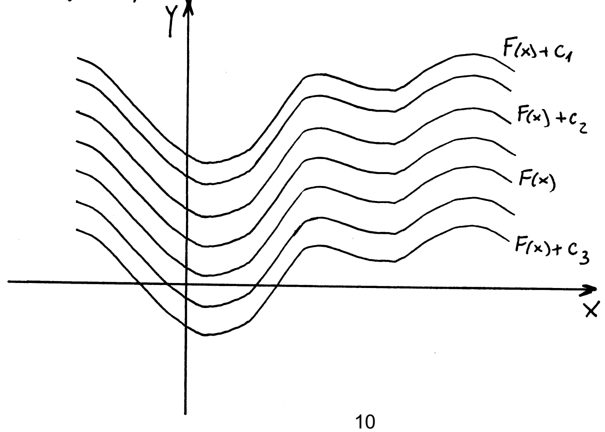
$$(F(x)+C)'=F'(x)=f(x).$$

Opšti izraz $F(x)+C$ skupa svih primitivnih f-ja f-je $f(x)$ zovemo neodređeni integral f-je $f(x)$ i označavamo ga sa znakom $\int$:

$$\int f(x)\,dx=F(x)+C \qquad \text{akko} \qquad d[F(x)+C]=f(x)\,dx$$

(ako i samo ako)

Geometriski, u xOy koordinatnom sistemu, grafici svih primitivnih f-ja date f-je $f(x)$ predstavljaju familiju krivih, koje zavise od parametra C, i koje se mogu izvesti jedna iz druge paralelnom translacijom duž Y-ose.



Osobine neodređenog integrala:

I $\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$ ili $d\int f(x)\,dx = f(x)\,dx$

(izvod integrala) (diferencijal integrala)

II $\int F'(x)\,dx = F(x)+C$ ili $\int dF(x) = F(x)+C$

III $\int a\,f(x)\,dx = a\int f(x)\,dx$ tj. konstantu a koja množi f-ju možemo izvesti ispred znaka integrala

IV $\int[f_1(x)+f_2(x)-f_3(x)]\,dx = \int f_1(x)\,dx+\int f_2(x)\,dx-\int f_3(x)\,dx$

tj. integral sume je jednak sumi integrala svih članova

Osnovne formule integriranja:

1. $\int u^a\,du = \frac{u^{a+1}}{a+1}+c,\ a\neq -1$
2. $\int u^{-1}\,du = \int\frac{du}{u} = \ln|u|+c$
3. $\int a^u\,du = \frac{a^u}{\ln a}+c$

$\int e^u\,du = e^u+c$

4. $\int \sin u\,du = -\cos u+c$
5. $\int \cos u\,du = \sin u+c$
6. $\int \frac{du}{\cos^2 u} = tg u+c$
7. $\int \frac{du}{\sin^2 u}\quad ctg u+c$
8. $\int \frac{du}{u^2+a^2} = \frac{1}{a}\,arctg\frac{u}{a}+c$
9. $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+c$
10. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a}+c$
11. $\int \frac{du}{\sqrt{u^2+a}} = \ln\left|u+\sqrt{u^2+a}\right|+c$

U osnovnim formulama integriranja a predstavlja konstantu, u je nezavisna promjenjiva ili bilo koja (diferencijabilna) f-ja neke nezavisne promjenjive. Navedimo nekoliko primjera korištenja osnovnih formula integriranja:

- Integral $I_1=\int\sqrt{x}\,dx=\int x^{\frac{1}{2}}dx$ predstavlja formulu 1 $\left(\int u^a\,du = \frac{u^{a+1}}{a+1}+c,\ a\neq-1\right)$ gdje su $u=x$, $a=\frac{1}{2}$. Prema toj formuli $I_1=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c=\frac{2}{3}\sqrt{x^3}+c$.
- Integral $I_2=\int 3^x\,dx$ predstavlja formulu 3 $\left(\int a^u\,du=\frac{a^u}{\ln a}+c\right)$ gdje su $a=3$, $u=x$. Prema toj formuli $I_2=\frac{3^x}{\ln 3}+c$.
- Integral $I_3=\int\frac{dt}{t^2+3}$ predstavlja formulu 8 $\left(\int\frac{du}{u^2+a^2}=\frac{1}{a}\,arctg\frac{u}{a}+c\right)$ gdje su $u=t$, $a=\sqrt{3}$. Prema toj formuli $I_3=\frac{1}{\sqrt{3}}\,arctg\frac{t}{\sqrt{3}}+c$.
- Integral $I_4=\int\frac{d\varphi}{\sqrt{\varphi^2-5}}$ predstavlja formulu 11 $\left(\int\frac{du}{\sqrt{u^2+a}}=\ln\left|u+\sqrt{u^2+a}\right|+c\right)$ gdje su $u=\varphi$, $a=-5$. Prema toj formuli $I_4=\ln\left|\varphi+\sqrt{\varphi^2-5}\right|+c$
- Integral $I_5=\int\frac{2x}{x^2+7}dx=\int\frac{(x^2+7)'}{x^2+7}dx=\int\frac{d(x^2+7)}{x^2+7}$ predstavlja formulu 2 $\left(\int\frac{du}{u}=\ln|u|+c\right)$ pri čemu je $u=x^2+7$ (zato što je $d(x^2+7)=2x\,dx$). Prema toj formuli $I_5=\ln(x^2+7)+c$. Znak apsolutne vrijednosti smo izostavili zato što je uvijek $x^2+7>0$.

U opštem slučaju, u formulama 2, 9 i 11

($\int \frac{du}{u} = \ln|u| + c$, $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + c$ i $\int \frac{du}{\sqrt{u^2+a}} = \ln|u+\sqrt{u^2+a}|+c$)

pišemo apsolutnu vrijednost samo u slučaju kada izraz ispod logaritma može imati negativnu vrijednost.

- Integral $I_6 = \int 5\sin 5t\,dt = \int \sin 5t\,d(5t)$ predstavlja formulu 4 ($\int \sin u\,du = -\cos u + c$) pri čemu je $u=5t$. Prema toj formuli $I_6 = -\cos 5t + c$.
- Integral $I_7 = \int e^{\sin\varphi}\cos\varphi\,d\varphi = \int e^{\sin\varphi}\,d\sin\varphi$ (zato što $d\sin\varphi = \cos\varphi\,d\varphi$) predstavlja formulu 3 ($\int e^u du = e^u + c$), pri čemu je $u=\sin\varphi$. Možemo zaključiti $I_7 = e^{\sin\varphi}+c$.
- Posmatrajmo integral $I_8 = \int\frac{e^x dx}{e^{2x}-1}$. Primjetimo da, zato što je $de^x = e^x dx$ možemo pisati $I_8 = \int \frac{e^x dx}{e^{2x}-1} = \int\frac{de^x}{(e^x)^2-1}$. Prema formuli 9 ($\int\frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u-a}\right|$) pri čemu je $u=e^x$, $a=1$ možemo zaključiti $I_8 = \frac{1}{2}\ln\frac{|e^x-1|}{e^x+1} + c$.

(#) Naći sljedeće integrale i provjeriti rezultat diferenciranjem

a) $\int\frac{dx}{x^3}$ b) $\int\frac{dx}{\sqrt{2-x^2}}$ c) $\int 3^t 5^t dt$ d) $\int\sqrt{y+1}\,dy$

e) $\int\frac{dx}{2x^2-6}$

Rj.: a) $\int\frac{dx}{x^3} = \int x^{-3}dx = \frac{x^{-2}}{-2} + c = C - \frac{1}{2x^2}$

Koristili smo formulu $\int u^{\alpha} du = \frac{u^{\alpha+1}}{\alpha+1}+c$ gdje je $u=x$, $\alpha=-3$.

Provjera:

Diferencirajmo dobijenu f-ju

$$d\left(C - \frac{1}{2x^2}\right) = -\frac{1}{2}(x^{-2})'dx = \left(-\frac{1}{2}\right)(-2)\,x^{-3}dx = x^{-3}dx$$
$$= \frac{dx}{x^3}$$

b) $\int\frac{dx}{\sqrt{2-x^2}} = \arcsin\frac{x}{\sqrt{2}} + c$

Koristili smo formulu $\int\frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a}+c$ gdje je $u=x$, $a=\sqrt{2}$

Provjera:

$d\left(\arcsin\frac{x}{\sqrt{2}}+c\right) =$

$$=\left(\arcsin\frac{x}{\sqrt{2}}\right)' dx = \frac{\left(\frac{x}{\sqrt{2}}\right)'}{\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}}\, dx = \frac{dx}{\sqrt{2-x^2}}.$$

c) $\int 3^t 5^t dt = \int 15^t = \frac{15^t}{\ln 15} + C$

Koristili smo formulu $\int a^u du = \frac{a^u}{\ln a} + C$
pri čemu je $a = 15$, $u = t$

Provjera: $d\left(\frac{15^t}{\ln 15} + C\right) = \frac{1}{\ln 15}\left(15^t\right)' dt = \frac{1}{\ln 15} 15^t \ln 15\, dt$

$= 15^t dt$

d) $\int \sqrt{y+1}\, dy = \int (y+1)^{\frac{1}{2}} d(y+1) = \frac{(y+1)^{\frac{3}{2}}}{\frac{3}{2}} + C =$

$= \frac{2}{3}\sqrt{(y+1)^3} + C$

Koristili smo formulu $\int u^\alpha du = \frac{u^{\alpha+1}}{\alpha+1} + C$ pri
čemu je $u = y+1$, $\alpha = \frac{1}{2}$ $(d(y+1) = dy)$

Provjera:
$d\left(\frac{2}{3}\sqrt{(y+1)^2} + C\right) = \frac{2}{3}\left(\sqrt{(y+1)^3}\right)' dy = \frac{2}{3}\cdot\frac{3}{2}(y+1)^{\frac{3}{2}-1} dy$

$= \sqrt{y+1}\, dy$

e) $\int \frac{dx}{2x^2-6} = \frac{1}{2}\int \frac{dx}{x^2-3} = \frac{1}{2}\cdot\frac{1}{2\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C$

Koristili smo formulu $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$,
pri čemu je $u = x$, $a = \sqrt{3}$

Provjera:

$d\left(\frac{1}{4\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right| + C\right) = \frac{1}{4\sqrt{3}}\left(\ln\frac{x-\sqrt{3}}{x+\sqrt{3}}\right)' dx$

$= \frac{1}{4\sqrt{3}}\left(\ln(x-\sqrt{3}) - \ln(x+\sqrt{3})\right)' dx =$

$= \frac{1}{4\sqrt{3}}\left(\frac{1}{x-\sqrt{3}} - \frac{1}{x+\sqrt{3}}\right) dx = \frac{1}{4\sqrt{3}}\cdot\frac{x+\sqrt{3}-x+\sqrt{3}}{x^2-3}\, dx$

$= \frac{dx}{2(x^2-3)}$

(#) Odrediti integrale

a) $\int \frac{dx}{\sqrt[3]{5x}}$ b) $\int \frac{dt}{\sqrt{3-4t^2}}$ c) $\int \cos 3\varphi \, d\varphi$

d) $\int e^{-\frac{x}{2}} dx$ e) $\int \sin(ax+b) dx$ f) $\int \frac{dx}{5x+4}$

Rješenje

a) $$\int \frac{dx}{\sqrt[3]{5x}} = \int (5x)^{-\frac{1}{3}} = 5^{-\frac{1}{3}} \int x^{-\frac{1}{3}} dx = \frac{1}{\sqrt[3]{5}} \cdot \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + C =$$

$$= \frac{3}{2\sqrt[3]{5}} x^{\frac{2}{3}} + C = \frac{3}{2\sqrt[3]{5}} \sqrt[3]{x^2} + C$$

Koristili smo formulu $\int u^{\alpha} du = \frac{u^{\alpha+1}}{\alpha+1}$ pri čemu je $u=x$, $\alpha = -\frac{1}{3}$.

b) $$\int \frac{dt}{\sqrt{3-4t^2}} = \int \frac{dt}{\sqrt{4(\frac{3}{4}-t^2)}} = \frac{1}{\sqrt{4}} \int \frac{dt}{\sqrt{\frac{3}{4}-t^2}} = \frac{1}{2} \cdot \arcsin \frac{t}{\sqrt{\frac{3}{4}}} + C$$

$$= \frac{1}{2} \arcsin \frac{2t}{\sqrt{3}} + C$$

Koristili smo folmulu $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin \frac{u}{a} + C$ gdje $u=t$, $a = \frac{\sqrt{3}}{2}$

Ovo smo mogli uradili i na drugi način

$$\int \frac{dt}{\sqrt{3-4t^2}} = \int \frac{dt}{\sqrt{3-(2t)^2}} = \left| \begin{array}{l} d(2t) = 2\,dt \\ dt = \frac{1}{2} d(2t) \end{array} \right| = \frac{1}{2} \int \frac{d(2t)}{\sqrt{3-(2t)^2}}$$

$$= \frac{1}{2} \arcsin \frac{2t}{\sqrt{3}} + C$$

Koristili smo istu formulu gdje je $u=2t$, $a=\sqrt{3}$.

c) $$\int \cos 3\varphi \, d\varphi = \left| \begin{array}{l} d(3\varphi) = 3\,d\varphi \\ d\varphi = \frac{1}{3} d(3\varphi) \end{array} \right| = \frac{1}{3} \int \cos 3\varphi \, d(3\varphi) = \frac{1}{3} \sin 3\varphi + C$$

Koristili smo formulu $\int \cos u \, du = \sin u + C$ gdje je $u = 3\varphi$.

d) $$\int e^{-\frac{x}{2}} dx = \left| \begin{array}{l} d(-\frac{x}{2}) = -\frac{1}{2} dx \\ dx = -2\, d(-\frac{x}{2}) \end{array} \right| = -2 \int e^{-\frac{x}{2}} d(-\tfrac{x}{2}) = -2\, e^{-\frac{x}{2}} + C$$

Koristili smo formulu $\int e^u du = e^u + C$ gdje $u = -\frac{x}{2}$.

e) $$\int \sin(ax+b)\, dx = \left| \begin{array}{l} d(ax+b) = a\,dx \\ dx = \frac{1}{a} d(ax+b) \end{array} \right| = \frac{1}{a} \int \sin(ax+b)\, d(ax+b) =$$

$$= -\frac{1}{a} \cos(ax+b) + C$$

Koristili smo formulu $\int \sin u \, du = -\cos u + C$ pri čemu je $u=ax+b$

f) $$\int \frac{dx}{5x+4} = \left| \begin{array}{l} d(5x+4) = 5\,dx \\ dx = \frac{1}{5} d(5x+4) \end{array} \right| = \frac{1}{5} \int \frac{d(5x+4)}{5x+4} = \frac{1}{5} \ln|5x+4| + C$$

Koristili smo formulu $\int \frac{du}{u} = \ln|u| + C$ pri čemu je $u = 5x+4$.

Odrediti integrale

a) $\int (3-2x)^7 dx$ b) $\int \frac{dx}{\cos^2(m-nx)}$ c) $\int tg\,\varphi\, d\varphi$

Rj.

a) $$\int (3-2x)^7 dx = \left| \begin{array}{r} d(3-2x) = -2\,dx \\ dx = -\frac{1}{2}\, d(3-2x) \end{array} \right| = -\frac{1}{2}\int (3-2x)^7 d(3-2x)$$

$$= -\frac{1}{2} \cdot \frac{(3-2x)^8}{8} + c = -\frac{1}{16}(3-2x)^8 + c$$

Koristili smo formulu $\int u^\alpha du = \frac{x^{\alpha+1}}{\alpha+1} + c$ pri čemu je $u = 3-2x$, $\alpha = 7$.

b) $$\int \frac{dx}{\cos^2(m-nx)} = \left| \begin{array}{r} d(m-nx) = -n\,dx \\ dx = \frac{-1}{n}\, d(m-nx) \end{array} \right| = -\frac{1}{n}\int \frac{d(m-nx)}{\cos^2(m-nx)} =$$

$$= -\frac{1}{n}\, tg\,(m-nx) + c$$

Koristili smo formulu $\int \frac{du}{\cos^2 u} = tg\,u + c$ gdje $u = m-nx$.

c) $$\int tg\,\varphi\, d\varphi = \int \frac{\sin\varphi}{\cos\varphi}\, d\varphi = \left| \begin{array}{r} d(\cos\varphi) = -\sin\varphi\, d\varphi \\ \sin\varphi\, d\varphi = -d(\cos\varphi) \end{array} \right| = -\int \frac{d(\cos\varphi)}{\cos\varphi}$$

$$= -\ln|\cos\varphi| + c$$

Koristili smo formulu $\int \frac{du}{u} = \ln|u| + c$ gdje $u = \cos\varphi$.

Zadaci za vježbu

Odrediti sljedeće integrale

①. $\int x^4 dx$ ②. $\int \sqrt[5]{t^2}\, dt$

③. $\int \frac{dy}{3y^2}$ ④. $\int \frac{dx}{x+3}$

⑤. $\int (\alpha-5)^8 d\alpha$ ⑥. $\int \frac{dx}{x^2+9}$

⑦. $\int \frac{dv}{\sqrt{v^2+7}}$ ⑧. $\int \frac{dz}{2z^2-4}$

⑨. $\int \frac{dx}{\sqrt{4-x^2}}$ ⑩. $\int \sin\frac{x}{3}\, dx$

⑪. $\int \frac{1}{\sin^2 2\varphi}\, d\varphi$ ⑫. $\int e^{4x} dx$

⑬. $\int \frac{3\,dt}{5^{2t}}$ ⑭. $\int \frac{dx}{2x+5}$

⑮. $\int \frac{dx}{(3x+2)^3}$ ⑯. $\int ctg\, x\, dx$

Rješenja:

1. $\frac{x^5}{5}$ 2. $\frac{5}{7}\sqrt[5]{t^7}$ 3. $-\frac{1}{3y}$ 4. $\ln|x+3|$ 5. $\frac{(\alpha-5)^9}{9}$

6. $\frac{1}{3}\, arc\,tg\,\frac{x}{3}$ 7. $\ln(v+\sqrt{v^2+7})$ 8. $\frac{1}{4\sqrt{2}} \ln\left|\frac{z-\sqrt{2}}{z+\sqrt{2}}\right|$.

9. $\arcsin\frac{x}{2}$ 10. $-3\cos\frac{x}{3}$ 11. $-\frac{1}{2}\, ctg\, 2\varphi$ 12. $\frac{1}{4}e^{4x}$

13. $-\frac{3 \cdot 5^{-2t}}{2\ln 5}$ 14. $\frac{\ln|2x+5|}{2}$ 15. $-\frac{1}{6(3x+2)^2}$ 16. $\ln|\sin x|$

Izabrani **Zadaci za vježbu** sa rješenjima (iz lekcije Osnovne formule integriranja)

1. Pomoću osnovnih tabličnih integrala i najjednostavnijih pravila integracije odrediti sljedeće integrale:

a) $\int \sqrt{x}\, dx$

Rj. $\int \sqrt{x}\, dx = \int x^{\frac{1}{2}} dx = \frac{x^{\frac{1}{2}+1}}{\frac{3}{2}} + C = \frac{2}{3} x^{\frac{3}{2}} + C = \frac{2}{3}\sqrt{x^3} + C = \frac{2}{3} x\sqrt{x} + C$

b) $\int \sqrt[m]{x^n}\, dx$

Rj. $\int \sqrt[m]{x^n}\, dx = \int x^{\frac{n}{m}} dx = \frac{x^{\frac{n}{m}+1}}{\frac{m+n}{m}} + C = \frac{m}{m+n}\, x^{\frac{m+n}{m}} + C = \frac{m}{m+n} \sqrt{x^{m+n}} + C =$

$= \frac{m}{m+n} \sqrt[m]{x^m \cdot x^n} + C = \frac{m}{m+n}\, x \sqrt[m]{x^n}$

c) $\int \frac{dx}{x^2}$

Rj. $\int \frac{dx}{x^2} = \int x^{-2} dx = \frac{x^{-2+1}}{-1} + C = -x^{-1} + C = -\frac{1}{x} + C = C - \frac{1}{x}$

d) $\int 10^x dx$

$\boxed{\ln 10 \approx 2{,}30258}$

Rj. $\int 10^x dx = \frac{10^x}{\ln 10} + C \approx 0{,}43429\, 10^x + C$

e) $\int a^x e^x dx$

Rj. $\int a^x e^x dx = \int (a \cdot e)^x dx = \frac{(a \cdot e)^x}{\ln(a \cdot e)} + C = \frac{a^x e^x}{\ln a + \ln e} + C = \frac{a^x e^x}{1 + \ln a} + C$

f) $\int \frac{dx}{2\sqrt{x}}$

Rj. $\int \frac{dx}{2\sqrt{x}} = \frac{1}{2}\int x^{-\frac{1}{2}} dx = \frac{1}{2} \cdot \frac{x^{-\frac{1}{2}+1}}{\frac{1}{2}} + C = \frac{1}{2} \cdot 2 \cdot x^{\frac{1}{2}} + C = \sqrt{x} + C$

g) $\int \frac{dh}{\sqrt{2gh}}$

Rj. $\int \frac{dh}{\sqrt{2gh}} = \int \frac{dh}{\sqrt{2g} \cdot \sqrt{h}} = \frac{1}{\sqrt{2g}} \int \frac{dh}{h^{\frac{1}{2}}} = \frac{1}{\sqrt{2g}} \int h^{-\frac{1}{2}} dh = \frac{1}{\sqrt{2g}} \cdot \frac{h^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= \frac{2}{\sqrt{2g}} \sqrt{h} + C = \sqrt{\frac{4h}{2g}} + C = \sqrt{\frac{2h}{g}} + C$

h) $\int 3{,}4\, x^{-0{,}17} dx$

Rj. $\int 3{,}4\, x^{-0{,}17} dx = 3{,}4 \frac{x^{-0{,}17+1}}{0{,}83} + C = \frac{3{,}4}{0{,}83} x^{0{,}83} + C \approx 4{,}096\, x^{0{,}83} + C$

i) $\int (1-2u)\, du = \int du - 2\int u\, du$

Rj. $\int (1-2u)\, du = \int du - \int 2u\, du = u - 2 \cdot \frac{u^2}{2} + C = u - u^2 + C$

j) $\int (\sqrt{x}+1)(x-\sqrt{x}+1)\, dx$

Rj. $\int (\sqrt{x}+1)(x-\sqrt{x}+1)\, dx = \int (x\sqrt{x} \underline{\underline{-x}} \underline{\underline{\underline{+\sqrt{x}}}} \underline{\underline{+x}} \underline{\underline{\underline{-\sqrt{x}}}} + 1)\, dx =$

$= \int (x^{\frac{3}{2}} + 1)\, dx = \frac{x^{\frac{3}{2}+1}}{\frac{5}{2}} + x + C = \frac{2}{5}\sqrt{x^5} + x + C = \frac{2}{5} x^2 \sqrt{x} + x + C$

k)^{v} $\int \frac{\sqrt{x} - x^3 e^x + x^2}{x^3}\, dx$ Rj. $C - \frac{2}{3x\sqrt{x}} - e^x + \ln|x|$

2. Integriranje pomoću razlaganja podintegralne f-je na dijelove

Ako podintegralna f-ja predstavlja algebarsku sumu nekoliko članova, tada, prema osobini IV ($\int [f_1(x)+f_2(x)-f_3(x)]dx = \int f_1(x)\,dx + \int f_2(x)\,dx - \int f_3(x)\,dx$) možemo integrirati svaki član posebno.

Koristeći ovo, mnoge integrale možemo svesti na sumu jednostavnijih integrala.

Odrediti integrale

a) $\int (3x^2-2x+5)\,dx$ b) $\int \frac{2x^2+x-1}{x^3}\,dx$ c) $\int (1+e^x)^2\,dx$

d) $\int \frac{2x+3}{x^2-5}\,dx$ e) $\int \frac{x^2}{x^2+1}\,dx$ f) $\int tg^2\varphi\, d\varphi$.

Rj:

a) $$\int(3x^2-2x+5)\,dx = \int 3x^2\,dx - \int 2x\,dx + \int 5\,dx =$$
$$= 3\int x^2 dx - 2\int x\,dx + 5\int dx = 3\,\frac{x^3}{3} - 2\cdot\frac{x^2}{2} + 5x + c$$
$$= x^3 - x^2 + 5x + C$$

b) $$\int \frac{2x^2+x-1}{x^3}\,dx = 2\int\frac{dx}{x} + \int x^{-2}dx - \int x^{-3}dx =$$
$$= 2\ln|x| + \frac{x^{-1}}{-1} + \frac{x^{-2}}{-2} + c = 2\ln|x| - \frac{1}{x} - \frac{1}{2x^2} + c$$

c) $$\int(1+e^x)^2\,dx = \int(1+2e^x+(e^x)^2)\,dx = \int dx + 2\int e^x dx + \int e^{2x}dx$$
$$= \left| \begin{matrix} d(2x) = 2\,dx \\ dx = \frac{1}{2}d(2x) \end{matrix} \right| = \int dx + 2\int e^x dx + \frac{1}{2}\int e^{2x}d(2x) =$$
$$= x + 2e^x + \frac{1}{2}e^{2x} + c$$

d) $$\int\frac{2x+3}{x^2-5}\,dx = \int\frac{2x}{x^2-5}\,dx + 3\int\frac{dx}{x^2-5} = \int\frac{d(x^2-5)}{x^2-5} + 3\int\frac{dx}{x^2-5}$$
$$= \ln|x^2-5| + \frac{3}{2\sqrt{5}}\ln\left|\frac{x-\sqrt{5}}{x+\sqrt{5}}\right| + C$$

e) $$\int \frac{x^2}{x^2+1}\,dx = \int \frac{(x^2+1)-1}{x^2+1}\,dx = \int \left(1 - \frac{1}{x^2+1}\right) dx =$$

$$= \int dx - \int \frac{dx}{x^2+1} = x - \operatorname{arc\,tg} x + c$$

f) $$\int \operatorname{tg}^2\varphi\, d\varphi = \int (\operatorname{tg}\varphi)^2\, d\varphi = \int \left(\frac{\sin\varphi}{\cos\varphi}\right)^2 d\varphi =$$

$$= \int \frac{\sin^2\varphi}{\cos^2\varphi}\, d\varphi = \int \frac{1-\cos^2\varphi}{\cos^2\varphi}\, d\varphi = \int \left(\frac{1}{\cos^2\varphi} - 1\right) d\varphi$$

$$= \operatorname{tg}\varphi - \varphi + c\,.$$

Zadaci za vježbu

Odrediti integrale

① $\int \left(2\sqrt[5]{x} - \sqrt[3]{2x} + 5\right) dx$

② $\int (\sin\varphi - \cos\varphi)^2\, d\varphi$

③** $\int \frac{x^2+1}{x^2-1}\, dx$

④ $\int \frac{5x^2-6x+1}{\sqrt{x}}\, dx$

⑤** $\int \frac{x^3}{x^2+6}\, dx$

⑥ $\int (\operatorname{tg} x + \operatorname{ctg} x)^2\, dx$

⑦ $\int (e^x - e^{-x})^3\, dx$

⑧* $\int \frac{x^2-2}{x+2}\, dx$

* Racionalni algebarski razlomak nazivamo svodljiv, ako je stepen polinoma u brojniku veći ili jednak stepenu polinoma u nazivniku.

** Ovdje, kao u rješenju zadatka $\int \frac{x^2}{x^2+1}\, dx$, podintegralni svodljiv razlomak napisati u nesvodljivom obliku.

Rj:

1. $\frac{5}{3}\, x\sqrt[5]{x} - \frac{3\sqrt[3]{2}}{4}\, x\sqrt[3]{x} + 5x$ 2. $\varphi + \frac{1}{2}\cos 2\varphi$

3. $x + \ln\left|\frac{x-1}{x+1}\right|$ 4. $2\sqrt{x}\,(x-1)^3$ 5. $\frac{x^2}{2} - 3\ln(x^2+6)$

6. $\operatorname{tg} x - \operatorname{ctg} x$ 7. $\frac{1}{2}(e^{2x} - e^{-2x}) - 2x$

8. $\frac{(x-2)^2}{2} + 2\ln|x+2|\,.$

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija pomoću razlaganja podintegralne funkcije na dijelove)

1. Pomoću osnovnih tabličnih integrala i najjednostavnijih pravila integracije odrediti sljedeće integrale:

a) $\int \frac{\sqrt{x}-x^3e^x+x^2}{x^3}\,dx$

Rj. $\int \frac{\sqrt{x}-x^3e^x+x^2}{x^3}\,dx = \int\left(\frac{\sqrt{x}}{x^3}-e^x+\frac{1}{x}\right)dx = \int\left(x^{-\frac{5}{2}}-e^x+\frac{1}{x}\right)dx$

$= \frac{x^{-\frac{5}{2}+1}}{-\frac{3}{2}} - e^x + \ln|x| + c = -\frac{2}{3}\cdot\frac{1}{\sqrt{x^3}} - e^x + \ln|x| + C =$

$= C - \frac{2}{3x\sqrt{x}} - e^x + \ln|x| +$

b) $\int(2x^{-1,2}+3x^{-0,8}-5x^{9,38})\,dx$

Rj. $\int(2x^{-1,2}+3x^{-9,8}-5x^{9,38})\,dx = 2\cdot\frac{x^{-1,2+1}}{-0,2}+3\cdot\frac{x^{-9,8+1}}{0,2}-5\cdot\frac{x^{0,38+1}}{1,38}+c$

$= -\frac{2}{0,2}x^{-0,2}+\frac{3}{0,2}x^{0,2}-\frac{5}{1,38}x^{1,38}+C = C-10x^{-0,2}+15x^{0,2}-3,62x^{1,38}$

c) $\int\left(\frac{1-z}{z}\right)^2 dz$

Rj. $\int\left(\frac{1-z}{z}\right)^2 dz = \int\left(\frac{1}{z}-1\right)^2 dz = \int\left(\frac{1}{z^2}-\frac{2}{z}+1\right)dz = \int\left(z^{-2}-2\,\frac{1}{z}+1\right)dz =$

$= \frac{z^{-1}}{-1} - 2\ln|z| + z + c = C - \frac{1}{z} - \ln z^2 + z$

d) $\int \frac{(1-x)^2}{x\sqrt{x}}\,dx$

Rj. $\int \frac{(1-x)^2}{x\sqrt{x}}\,dx = \int \frac{1-2x+x^2}{x^{\frac{3}{2}}}\,dx = \int\left(x^{-\frac{3}{2}}-2x^{-\frac{1}{2}}+x^{\frac{1}{2}}\right)dx =$

$= \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} - 2\,\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = -2\cdot\frac{1}{x^{\frac{1}{2}}} - 2\cdot 2\sqrt{x} + \frac{2}{3}\sqrt{x^3} + C = \frac{-2}{\sqrt{x}} - 4\sqrt{x} + \frac{2}{3}x\sqrt{x} + C$

$= \frac{-2\cdot 3 - 4\sqrt{x}\cdot 3\sqrt{x} + 2x\sqrt{x}\cdot\sqrt{x}}{3\sqrt{x}} + C = \frac{2x^2-12x-6}{3\sqrt{x}} + C$

e) $\int \frac{(1+\sqrt{x})^3}{\sqrt[3]{x}}\,dx$

$\frac{1}{2}-\frac{1}{9}=\frac{9-2}{18}=\frac{7}{18}$ $\quad \frac{7}{18}-\frac{4}{18}=\frac{3}{18}=\frac{1}{6}$ $\quad \frac{12}{18}=\frac{2}{3}$

Rj. $\int \frac{(1+\sqrt{x})^3}{\sqrt[3]{x}}\,dx = \int\left(\frac{1+\sqrt{x}}{x^{\frac{1}{3}}}\right)^3 dx = \int\left(x^{-\frac{1}{3}}+x^{\frac{7}{18}}\right)^3 dx = \int\left(x^{-\frac{1}{3}}+3\cdot x^{-\frac{2}{3}}\cdot x^{\frac{7}{18}}+3x^{-\frac{1}{3}}\cdot x^{\frac{14}{18}}+\right.$

$\left.+x^{\frac{7}{6}}\right)dx = \int\left(x^{-\frac{1}{3}}+3x^{\frac{1}{6}}+3x^{\frac{2}{3}}+x^{\frac{7}{6}}\right)dx = \frac{x^{\frac{2}{3}}}{\frac{2}{3}}+3\cdot\frac{x^{\frac{7}{6}}}{\frac{7}{6}}+3\,\frac{x^{\frac{5}{3}}}{\frac{5}{3}}+\frac{x^{\frac{13}{6}}}{\frac{13}{6}}+C$

$= \frac{3}{2}\sqrt[3]{x^2}+\frac{18}{7}\sqrt[6]{x^7}+\frac{9}{5}\sqrt[3]{x^5}+\frac{6}{13}\sqrt[6]{x^{13}}+C =$

$= \frac{3}{2}\sqrt[3]{x^2}+\frac{18}{7}x\sqrt[6]{x}+\frac{9}{5}x\sqrt[3]{x^2}+\frac{6}{13}x^2\sqrt[6]{x}+C$

f) $\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}\,dx$

$\frac{2}{3}-\frac{1}{2}=\frac{4-3}{6}=\frac{1}{6}$ $\quad \frac{1}{4}-\frac{1}{2}=\frac{1-2}{4}=-\frac{1}{4}$

Rj. $\int \frac{\sqrt[3]{x^2}-\sqrt[4]{x}}{\sqrt{x}}\,dx = \int \frac{x^{\frac{2}{3}}-x^{\frac{1}{4}}}{x^{\frac{1}{2}}}\,dx = \int\left(x^{\frac{1}{6}}-x^{-\frac{1}{4}}\right)dx = \frac{x^{\frac{7}{6}}}{\frac{7}{6}}-\frac{x^{\frac{3}{4}}}{\frac{3}{4}}+C$

$= \frac{6}{7}\sqrt[6]{x^7}-\frac{4}{3}\sqrt[4]{x^3}+C = \frac{6}{7}x\sqrt[6]{x}-\frac{4}{3}\sqrt[4]{x^3}+C$

g) $\int \frac{dx}{\sqrt{3-3x^2}}$

Rj. $\int \frac{dx}{\sqrt{3-3x^2}} = \int \frac{dx}{\sqrt{3(1-x^2)}} = \int \frac{dx}{\sqrt{3}\cdot\sqrt{1-x^2}} = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{1-x^2}} = \frac{\sqrt{3}}{3}\arcsin x + C$

h) $\int \frac{3\cdot 2^x-2\cdot 3^x}{2^x}\,dx$

Rj. $\int \frac{3\cdot 2^x-2\cdot 3^x}{2^x}\,dx = \int\left(3-2\left(\frac{3}{2}\right)^x\right)dx = 3x-2\cdot\frac{\left(\frac{3}{2}\right)^x}{\ln\frac{3}{2}}+C = 3x-\frac{2\cdot 1,5^x}{\ln 1,5}+C$

i) $\int \frac{1+\cos^2x}{1+\cos 2x}\,dx$

Rj. $\int \frac{1+\cos^2x}{1+\cos 2x}\,dx = \int \frac{1+\cos^2x}{\underbrace{\sin^2x+\cos^2x}_{1}+\underbrace{\cos^2x-\sin^2x}_{\cos 2x}}\,dx = \int \frac{1+\cos^2x}{2\cos^2x}\,dx$

$= \int\left(\frac{1}{2\cos^2x}+\frac{\cos^2x}{2\cos^2x}\right)dx = \frac{1}{2}\int\left(\frac{1}{\cos^2x}+1\right)dx = \frac{1}{2}\operatorname{tg}x+\frac{1}{2}x+C$

2. Pomoću osnovnih tabličnih integrala i najjednostavnijih pravila integracije odrediti sljedeće integrale:

a) $\int \frac{\cos 2x}{\cos^2 x \sin^2 x} dx$

Rj. $\int \frac{\cos 2x}{\cos^2 x \cdot \sin^2 x} dx = \int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \cdot \sin^2 x} dx = \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} \right) dx =$

$= -\mathrm{ctg}\, x - \mathrm{tg}\, x + c = c - \mathrm{ctg}\, x - \mathrm{tg}\, x$

b) $\int \mathrm{tg}^2 x \, dx$

Rj. $\int \mathrm{tg}^2 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} dx = \int \frac{1-\cos^2 x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} - 1 \right) dx = \mathrm{tg}\, x - x + c$

c) $\int \mathrm{ctg}^2 x \, dx$

Rj. $\int \mathrm{ctg}^2 x \, dx = \int \left(\frac{\cos x}{\sin x} \right)^2 dx = \int \frac{\cos^2 x}{\sin^2 x} dx = \int \frac{1-\sin^2 x}{\sin^2 x} dx = \int \left(\frac{1}{\sin^2 x} - 1 \right) dx = c - \mathrm{ctg}\, x - x$

d) $\int 2 \sin^2 \frac{x}{2} dx$

$\left. \begin{array}{l} 1 = \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \\ \cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \end{array} \right\} - \Rightarrow 1 - \cos x = 2 \sin^2 \frac{x}{2}$

Rj. $\int 2 \sin^2 \frac{x}{2} dx = \int (1 - \cos x) dx = x - \sin x + c$

e) $\int \frac{(1+2x^2)\, dx}{x^2 (1+x^2)}$

Rj. $\int \frac{(1+2x^2)}{x^2(1+x^2)} dx = \int \frac{1+x^2+x^2}{x^2(1+x^2)} dx = \int \left[\frac{1+x^2}{x^2(1+x^2)} + \frac{x^2}{x^2(1+x^2)} \right] dx = \int \left(\frac{1}{x^2} + \frac{1}{1+x^2} \right) dx =$

$= \frac{x^{-1}}{-1} + \mathrm{arctg}\, x + c = c - \frac{1}{x} + \mathrm{arctg}\, x$

f) $\int \frac{(1+x)^2 dx}{x(1+x^2)}$

Rj. $\int \frac{(1+x)^2 dx}{x(1+x^2)} = \int \frac{1+2x+x^2}{x(1+x^2)} dx = \int \left(\frac{1+x^2}{x(1+x^2)} + \frac{2x}{x(1+x^2)} \right) dx = \int \left(\frac{1}{x} + \frac{2}{1+x^2} \right) dx =$

$= \ln|x| + 2\,\mathrm{arctg}\, x + c$

g) $\int \frac{dx}{\cos 2x + \sin^2 x}$

Rj. $\int \frac{dx}{\cos 2x + \sin^2 x} = \int \frac{dx}{\cos^2 x - \sin^2 x + \sin^2 x} = \int \frac{dx}{\cos^2 x} = \mathrm{tg}\, x + c$

h) $\int (\arcsin x + \arccos x)\, dx$

Rj. $\int (\arcsin x + \arccos x)\, dx = \int \left(\alpha - \alpha + \frac{\pi}{2} \right) dx = \frac{\pi}{2} x + c$

$\sin \alpha = x$, α je neki ugao
$x \in [-1, 1]$

$\Rightarrow \arcsin x + \arccos x = \frac{\pi}{2}$

$\arcsin x = \alpha$

$\cos\left(-\alpha + \frac{\pi}{2}\right) = \cos(-\alpha) \cos \frac{\pi}{2} - \sin(-\alpha) \sin \frac{\pi}{2} = -\sin(-\alpha) = \sin \alpha$

tj. $\cos\left(-\alpha + \frac{\pi}{2}\right) = \sin \alpha = x \quad \Rightarrow \quad \arccos x = -\alpha + \frac{\pi}{2}$

Odredite sljedeće integrale:

③ $\int(2x^3+5x^2-7x-6)dx=\int 2x^3dx+\int 5x^2dx-\int 7x\,dx-\int 6\,dx=$

$=2\int x^3dx+5\int x^2dx-7\int x\,dx-6\int dx=2\cdot\frac{x^4}{4}+5\cdot\frac{x^3}{3}-7\cdot\frac{x^2}{2}-6x+C$

$=\frac{x^4}{2}+\frac{5x^3}{3}-\frac{7x^2}{2}-6x+C$

④ $\int\frac{5x^7+2x^5-x+6}{x^3}dx=\int\left(\frac{5x^7}{x^3}+\frac{2x^5}{x^3}-\frac{x}{x^3}+\frac{6}{x^3}\right)dx=$

$5\int x^4dx+2\int x^2dx-\int x^{-2}dx+6\int x^{-3}dx=5\cdot\frac{x^5}{5}+2\cdot\frac{x^3}{3}-\frac{x^{-1}}{-1}+6\cdot\frac{x^{-2}}{-2}+C$

$=x^5+\frac{2x^3}{3}+\frac{1}{x}-3\cdot\frac{1}{x^2}+C$

⑤ $\int\sqrt{x\sqrt[3]{x\sqrt{x}}}\,dx=\int\sqrt{x\sqrt[3]{\sqrt{x^2\cdot x}}}\,dx=\int\sqrt{x\sqrt[6]{x^3}}\,dx$

$=\int\sqrt{\sqrt[6]{x^6\cdot x^3}}\,dx=\int\sqrt[12]{x^9}\,dx=\int x^{\frac{9}{12}}dx=\int x^{\frac{3}{4}}dx$

$=\frac{x^{\frac{7}{4}}}{\frac{7}{4}}+C=\frac{4}{7}\sqrt[4]{x^7}+C$

⑥ $\int(x^2+\sqrt{x})^2dx=\int(x^4+2x^2\sqrt{x}+x)dx=\int x^4dx+2\int\underbrace{x^2\cdot x^{\frac{1}{2}}}_{x^{\frac{5}{2}}}dx+\int x\,dx$

$=\frac{x^5}{5}+2\cdot\frac{x^{\frac{7}{2}}}{\frac{7}{2}}+\frac{x^2}{2}+C=\frac{x^5}{5}+\frac{4}{7}\sqrt{x^7}+\frac{x^2}{2}+C$

⑦ $\int\frac{x}{x+1}dx=\int\frac{x+1-1}{x+1}dx=\int\left(1-\frac{1}{x+1}\right)dx=$

$=\int dx-\int\frac{dx}{x+1}=x-\ln|x+1|+C$

⑧ $\int\frac{x^2}{x-1}dx=\int\frac{x^2-1+1}{x-1}dx=\int\frac{x^2-1}{x-1}dx+\int\frac{1}{x-1}dx=$

$=\int\frac{(x-1)(x+1)}{(x-1)}dx+\int\frac{1}{x-1}dx=\int(x+1)dx+\int\frac{1}{x-1}dx=\frac{x^2}{2}+x+\ln|x-1|+C$

⑨ $\int\frac{x^2}{x+2}dx=\int\frac{x^2+4-4}{x+2}dx=\int\left(\frac{x^2-4}{x+2}+\frac{4}{x+2}\right)dx=$

$=\int\frac{(x-2)(x+2)}{x+2}dx+4\int\frac{dx}{x+2}=\int(x-2)dx+4\int\frac{dx}{x+2}=\frac{x^2}{2}-2x+4\ln|x+2|+C$

II način bi bio da podjelimo x^2 sa $x+2$ pa izvadimo integral od dobijenog rezultata $x^2:(x+2)=x-2+\frac{4}{x+2}$ $\quad\int\frac{x^2}{x+2}dx=\int\left(x-2+\frac{4}{x+2}\right)dx$

⑩ $\int\frac{x^3}{x-3}dx=\int\frac{x^3-27+27}{x-3}dx=\int\frac{x^3-27}{x-3}dx+\int\frac{27}{x-3}dx=$

$=\int\frac{(x-3)(x^2+3x+9)}{x-3}dx+\int\frac{27}{x-3}dx=\int(x^2+3x+9)dx+27\int\frac{dx}{x-3}$

$=\frac{x^3}{3}+\frac{3x^2}{2}+9x+27\ln|x-3|+C$

⑪ $\int tg^2x\,dx=\int\frac{\sin^2x}{\cos^2x}dx=\int\frac{1-\cos^2x}{\cos^2x}dx=\int\frac{1}{\cos^2x}dx-\int dx$

$=tg\,x-x+C$

⑫ ✓ $\int\frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}dx$ $\qquad$ Rj. $-\frac{24}{17}\sqrt[12]{x^{17}}+\frac{4}{5}\sqrt[4]{x^5}+\frac{4}{3}\sqrt[4]{x^3}+C$

⑬ ✓ $\int\frac{e^{3x}+1}{e^x+1}dx$ $\qquad$ Rj. $\frac{1}{2}e^{2x}-e^x+x+C$

⑭ ✓ $\int\frac{1}{\sin^2 2x}dx$ $\qquad$ Rj. $-\frac{1}{2}\cdot\frac{\cos 2x}{\sin 2x}+C$

3. Integracija pomoću smjene promjenjivih

Veoma efikasna metoda integriranja je metoda pomoću smjene promjenjivih, a rezultat metode je da se dati integral zamjeni drugim integralom.

Posmatrajmo dati integral $\int f(x)\,dx$. Ako je moguće, želimo promjenjivu x zamjeniti nekom novom promjenjivom t, koristeći smjenu $x=\varphi(t)$. Tada je $dx=\varphi'(t)\,dt$ pa imamo

$$\int f(x)\,dx=\int f[\varphi(t)]\,\varphi'(t)\,dt=\int F(t)\,dt.$$

Na ovaj način se često zadani integral svodi na elementarni tablični integral. Na primjer, želimo odrediti integral $J=\int\frac{dx}{1+\sqrt{x}}$, uvođenjem smjene $x=t^2$. Tada je $dx=2t\,dt$ pa imamo

$$J=\int\frac{2t\,dt}{1+t}=2\int\frac{t+1-1}{t+1}\,dt=2\int\left(1-\frac{1}{t+1}\right)dt=$$

$$=2\int dt-2\int\frac{dt}{t+1}=2t-2\ln(t+1)+C=$$

$$=2\sqrt{x}-2\ln(\sqrt{x}+1)+C$$

Isti integral smo mogli odrediti i uvođenjem smjene $t=1+\sqrt{x}$ iz čega slijedi $x=(t-1)^2$, $dx=2(t-1)\,dt$

$$J=\int\frac{2(t-1)\,dt}{t}=2\int\left(1-\frac{1}{t}\right)dt=2t-2\ln t+C=2(1+\sqrt{x})-2\ln(1+\sqrt{x})+C$$

(#) Odrediti integrale

a) $\int\frac{2x\,dx}{x^4+3}$ b) $\int\frac{\sin x\,dx}{\sqrt{1+2\cos x}}$ c) $\int\frac{x\,dx}{\sqrt[3]{x^2+a}}$

d) $\int\frac{\sqrt{1+\ln x}}{x}\,dx$ e) $\int\frac{dy}{\sqrt{e^y+1}}$ f) $\int\frac{dt}{\sqrt{(1-t^2)^3}}$

Rj.

a) $$\int\frac{2x\,dx}{x^4+3}=\left|\begin{array}{l}x^2=t\\2x\,dx=dt\end{array}\right|=\int\frac{dt}{t^2+3}=\frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{t}{\sqrt{3}}+C=$$

$$=\frac{1}{\sqrt{3}}\,\mathrm{arctg}\,\frac{x^2}{\sqrt{3}}+C$$

b) $$\int\frac{\sin x\,dx}{\sqrt{1+2\cos x}}=\left|\begin{array}{l}1+2\cos x=t\\-2\sin x\,dx=dt\\\sin x\,dx=-\frac{1}{2}\,dt\end{array}\right|=-\frac{1}{2}\int\frac{dt}{\sqrt{t}}=-\frac{1}{2}\int t^{-1/2}\,dt$$

$$=-\frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}}+C=C-\sqrt{t}=C-\sqrt{1+2\cos x}$$

c) $$\int\frac{x\,dx}{\sqrt[3]{x^2+a}}=\left|\begin{array}{l}x^2+a=z\\2x\,dx=dz\\x\,dx=\frac{1}{2}\,dz\end{array}\right|=\frac{1}{2}\int\frac{dz}{\sqrt[3]{z}}=\frac{1}{2}\int z^{-\frac{1}{3}}\,dz=$$

$$=\frac{1}{2}\cdot\frac{z^{\frac{2}{3}}}{\frac{2}{3}}+C=\frac{1}{2}\cdot\frac{3}{2}\cdot\sqrt[3]{z^2}+C=\frac{3}{4}\sqrt[3]{(x^2+a)^2}+C$$

d) $$\int\frac{\sqrt{1+\ln x}}{x}\,dx=\left|\begin{array}{l}1+\ln x=v\\\frac{1}{x}\,dx=dv\end{array}\right|=\int\sqrt{v}\,dv=\int v^{\frac{1}{2}}\,dv=\frac{v^{\frac{3}{2}}}{\frac{3}{2}}+C$$

$$=\frac{2}{3}\sqrt{v^3}+C=\frac{2}{3}\sqrt{(1+\ln x)^3}+C$$

e) $$\int \frac{dy}{\sqrt{e^y+1}} = \left| \begin{array}{ll} e^y+1=t^2 & (t^2-1)\,dy = 2t\,dt \\ e^y = t^2-1 & dy = \dfrac{2t\,dt}{t^2-1} \\ de^y = d(t^2-1) & \\ e^y dy = 2t\,dt & \end{array} \right| =$$

$$= \int \frac{\frac{2t\,dt}{t^2-1}}{\sqrt{t^2}} = \int \frac{2t\,dt}{t(t^2-1)} = 2\int \frac{dt}{t^2-1} = 2\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$$

$$= \ln \frac{\sqrt{e^y+1}-1}{\sqrt{e^y+1}+1} + C$$

f) $$\int \frac{dt}{\sqrt{(1-t^2)^3}} = \left| \begin{array}{l} t = \sin\varphi \\ dt = \cos\varphi\, d\varphi \\ 1-t^2 = 1-\sin^2\varphi = \cos^2\varphi \end{array} \right| =$$

$$= \int \frac{\cos\varphi\, d\varphi}{\sqrt{\cos^6\varphi}} = \int \frac{\cos\varphi\, d\varphi}{\cos^3\varphi} = \int \frac{d\varphi}{\cos^2\varphi} = tg\,\varphi + C$$

$$= \frac{\sin\varphi}{\cos\varphi} + C = \frac{\sin\varphi}{\sqrt{1-\sin^2\varphi}} + C = \frac{t}{\sqrt{1-t^2}} + C$$

Zadaci za vježbu

Izračunati sljedeće integrale i provjeriti rezultat diferenciranjem:

① $\int \frac{x^2\,dx}{5-x^6}$. Pomoću smjene $t=x^3$.

② $\int \frac{e^x\,dx}{3+4e^x}$. Pomoću smjene $z=3+4e^x$.

③ $\int tg^3\varphi\, d\varphi$. Pomoću smjene $\varphi = arc\,tg\, t$.

④ $\int x^3\sqrt{a-x^2}\,dx$. Pomoću smjene $\sqrt{a-x^2}=z$.

⑤ $\int \frac{x^2-x}{(x-2)^3}\,dx$. Pomoću smjene $x-2=t$

⑥ $\int x\sqrt{a-x}\,dx$, Pomoću smjene $a-x=t^2$

⑦* $\int \frac{dx}{x\sqrt{1+x^2}}$. Pomoću smjene $x=\frac{1}{t}$.

⑧* $\int \frac{dx}{\sin 2x}$. Pomoću smjene $tg\,x = z$.

Odrediti integrale

⑨ $\int \frac{x\,dx}{\sqrt{x^4+1}}$ ⑩ $\int \frac{\sqrt{x}\,dx}{1+\sqrt{x}}$

⑪ $\int \frac{e^{2x}\,dx}{e^x-1}$ ⑫ $\int \frac{dx}{x\ln x}$

⑬ $$\int \frac{\cos x\, dx}{\sqrt{1+2\sin^2 x}}$$

⑭ $$\int \frac{\sin 2x\, dx}{\sqrt{2+\cos^2 x}}$$

⑮* $$\int \frac{e^{2x}\, dx}{\sqrt[4]{1+e^x}}$$

⑯* $$\int \frac{\sqrt{x}\, dx}{1+\sqrt[4]{x^3}}$$

Rješenja:

1. $\frac{1}{6\sqrt{5}} \ln\left|\frac{x^3+\sqrt{5}}{x^3-\sqrt{5}}\right|$

2. $\frac{1}{4}\ln(3+4e^x)$

3. $\frac{1}{2} tg^2\varphi + \ln|\cos\varphi|$

4. $-\frac{3x^2+2a}{15}\sqrt{(a-x^2)^2}$

5. $\ln|x-2| - \frac{3x-5}{(x-2)^2}$

6. $\frac{2}{15}(3x^2-ax-2a^2)\sqrt{a-x}$

7. $\pm \ln \frac{x}{1\pm\sqrt{1+x^2}}$, gdje je "+" ako $x>0$, "−" ako je $x<0$

ili drugačije $\ln \frac{|x|}{1+}$

8. $\frac{1}{2}\ln|tg\, x|$

9. $\frac{1}{2}\ln(x^2+\sqrt{1+x^4}$

10. $x-2\sqrt{x}+2\ln(1+\sqrt{x})$

11. $e^x+\ln|e^x-1|$

12. $\ln|\ln x|$

13. $\frac{1}{\sqrt{2}}\ln\left(\sin x+\sqrt{\frac{1}{2}+\sin^2 x}\right)$

14. $-2\sqrt{2+\cos^2 x}$

15. $\frac{4}{21}(3e^x-4)\sqrt[4]{(e^x+1)^3}$

16. $\frac{4}{3}\left(\sqrt[4]{x^3}-\ln\left(1+\sqrt[4]{x^3}\right)\right)$.

Izabrani **Zadaci za vježbu** sa rješenjima
(iz lekcije Integracija pomoću zamjene promjenjivih)

1. $$\int \frac{dx}{2x+5} = \left|\begin{matrix} t=2x+5 \\ dt=2dx \\ dx=\frac{dt}{2}\end{matrix}\right| = \int \frac{\frac{dt}{2}}{t} = \frac{1}{2}\int\frac{dt}{t} = \frac{1}{2}\ln|t|+c = \frac{1}{2}\ln|2x+5|+c$$

2. $$\int \sin(4x+1)\,dx = \left|\begin{matrix} 4x+1=t \\ 4dx=dt \\ dx=\frac{dt}{4}\end{matrix}\right| = \int \sin t\cdot\frac{dt}{4} = \frac{1}{4}\int \sin t\,dt = -\frac{1}{4}\cos t + c =$$
$$= -\frac{1}{4}\cos(4x+1)+c$$

3. $$\int (3x-1)^9\,dx = \left|\begin{matrix} 3x-1=t \\ 3dx=dt \\ dx=\frac{dt}{3}\end{matrix}\right| = \int t^9\cdot\frac{dt}{3} = \frac{1}{3}\int t^9 dt = \frac{1}{3}\cdot\frac{t^{10}}{10}+c =$$
$$= \frac{(3x-1)^{10}}{30}+c$$

4. $$\int e^{1-3x}\,dx = \left|\begin{matrix} 1-3x=t \\ -3dx=dt \\ dx=-\frac{dt}{3}\end{matrix}\right| = \int e^t\cdot -\frac{dt}{3} = -\frac{1}{3}\int e^t dt = -\frac{1}{3}e^t+c =$$
$$-\frac{1}{3}e^{1-3x}+c$$

5. $$\int \frac{dx}{\sqrt{1-(3x+2)^2}} = \left|\begin{matrix} 3x+2=t \\ 3dx=dt \\ dx=\frac{dt}{3}\end{matrix}\right| = \frac{1}{3}\int\frac{dt}{\sqrt{1-t^2}} = \frac{1}{3}\arcsin t + c =$$
$$= \frac{1}{3}\arcsin(3x+2)+c$$

6.ᵛ $\int \cos(6x+4)\,dx$ Rj. $\frac{1}{6}\sin(6x+4)+c$

7.ᵛ $\int \frac{dx}{\cos^2(7x+8)}$ Rj. $\frac{1}{7}\cdot\frac{\sin(7x+8)}{\cos(7x+8)}$

8.ᵛ $\int \frac{dx}{1+(5x-2)^2}$ Rj. $\frac{1}{5}\,\mathrm{arc\,tg}(5x-2)$

9.
$$\int \frac{dx}{4x^2+9} = \int \frac{dx}{(2x)^2+3^2} = \left| \begin{array}{l} 2x = 3t \\ 2dx = 3dt \\ dx = \frac{3}{2}dt \\ t = \frac{2x}{3} \end{array} \right| = \frac{3}{2} \int \frac{dt}{(3t)^2+3^2} = \frac{3}{2} \int \frac{dt}{9t^2+9} =$$
$$= \frac{3}{2} \cdot \frac{1}{9} \int \frac{dt}{t^2+1} = \frac{1}{6} \operatorname{arctg} t + C = \frac{1}{6} \operatorname{arctg} \frac{2x}{3} + C$$

10.
$$\int \frac{dx}{\sqrt{2x^2+25}} = \int \frac{dx}{\sqrt{(\sqrt{2}x)^2+5^2}} = \left| \begin{array}{l} \sqrt{2}x = 5t \\ \sqrt{2}\,dx = 5dt \\ dx = \frac{5}{\sqrt{2}}\,dt \\ t = \frac{\sqrt{2}}{5}x \end{array} \right| = \frac{5}{\sqrt{2}} \int \frac{dt}{\sqrt{\underbrace{25t^2+25}_{25\cdot(t^2+1)}}} =$$
$$= \frac{5}{\sqrt{2}} \cdot \frac{1}{5} \int \frac{dt}{\sqrt{t^2+1}} = \frac{1}{\sqrt{2}} \cdot \ln \left| t + \sqrt{t^2+1} \right| + C = \frac{\sqrt{2}}{2} \ln \left| \frac{\sqrt{2}}{5}x + \sqrt{\frac{2}{25}x^2+1} \right| + C$$

11.
$$\int \frac{dx}{5x^2-49} = \int \frac{dx}{(\sqrt{5}x)^2-7^2} = \left| \begin{array}{l} \sqrt{5}x = 7t \\ \sqrt{5}dx = 7dt \\ dx = \frac{7}{\sqrt{5}}\,dt \\ t = \frac{\sqrt{5}x}{7} \end{array} \right| = \frac{7}{\sqrt{5}} \int \frac{dt}{49t^2-49} = \frac{7}{\sqrt{5}} \cdot \frac{1}{49} \int \frac{dt}{t^2-1}$$
$$= \frac{1}{7\sqrt{5}} \cdot \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| + C = \frac{1}{14\sqrt{5}} \ln \left| \frac{\frac{\sqrt{5}}{7}x-1}{\frac{\sqrt{5}}{7}x+1} \right| + C = \frac{1}{14\sqrt{5}} \ln \left| \frac{\sqrt{5}x-7}{\sqrt{5}x+7} \right|$$

12.
$$\int \frac{dx}{\sqrt{7-9x^2}} = \int \frac{dx}{\sqrt{(\sqrt{7})^2-(3x)^2}} = \left| \begin{array}{l} 3x = \sqrt{7}t \\ 3dx = \sqrt{7}\,dt \\ dx = \frac{\sqrt{7}}{3}\,dt \\ t = \frac{3x}{\sqrt{7}} \end{array} \right| = \frac{\sqrt{7}}{3} \int \frac{dt}{\sqrt{7-7t^2}} = \frac{\sqrt{7}}{3} \cdot \frac{1}{\sqrt{7}} \int \frac{dt}{\sqrt{1-t^2}}$$
$$= \frac{1}{3} \arcsin t + C = \frac{1}{3} \arcsin\left(\frac{3x}{\sqrt{7}}\right) + C$$

13.ˇ $\int \frac{dx}{4x^2+11}$, Rj. $\frac{\sqrt{11}}{22} \operatorname{arctg} \frac{2\sqrt{11}x}{11} + C$

14.ˇ $\int \frac{dx}{\sqrt{9x^2-16}}$, Rj. $\frac{1}{3} \ln \left| 3x + \sqrt{9x^2-16} \right| + C$

15.ˇ $\int \frac{dx}{\sqrt{2x^2+5}}$, Rj. $\frac{\sqrt{2}}{2} \ln \left| \frac{\sqrt{10}}{5}x + \sqrt{\frac{2}{5}x^2+1} \right| + C$

16.
$$\int \operatorname{ctg} x\, dx = \int \frac{\cos x}{\sin x}\, dx = \left| \begin{array}{l} \sin x = t \\ \cos x\, dx = dt \end{array} \right| = \int \frac{dt}{t} = \ln|t| + C =$$
$$= \ln|\sin x| + C$$

17.
$$\int \frac{3x^2+4x-4}{x^3+2x^2-4x+6}\, dx = \left| \begin{array}{l} x^3+2x^2-4x+6 = t \\ (3x^2+4x-4)\, dx = dt \end{array} \right| = \int \frac{dt}{t} = \ln|t| + C =$$
$$= \ln|x^3+2x^2-4x+6| + C$$

18.
$$\int \frac{x-5}{\sqrt{x^2-10x+7}}\, dx = \left| \begin{array}{l} x^2-10x+7 = t \\ (2x-10)\, dx = dt \\ (x-5)\, dx = \frac{dt}{2} \end{array} \right| = \frac{1}{2} \int \frac{dt}{\sqrt{t}} = \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C =$$
$$= \frac{1}{2} \cdot 2\sqrt{t} + C = \sqrt{x^2-10x+7} + C$$

19.
$$\int \frac{x-3}{x^2-6x+7}\, dx = \left| \begin{array}{l} x^2-6x+7 = t \\ (2x-6)\, dx = dt \\ (x-3)\, dx = \frac{dt}{2} \end{array} \right| = \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln|t| + C$$
$$= \frac{1}{2} \ln|x^2-6x+7| + C$$

20.
$$\int \frac{x^3\, dx}{\sqrt{x^4+1}} = \left| \begin{array}{l} x^4+1 = t \\ 4x^3 dx = dt \\ x^3 dx = \frac{dt}{4} \end{array} \right| = \frac{1}{4} \int \frac{dt}{\sqrt{t}} = \frac{1}{4} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{1}{4} \cdot 2\sqrt{t} + C$$
$$= \frac{1}{2} \sqrt{x^4+1} + C$$

21.ˇ $\int \frac{3x^2}{\sqrt{x^3-2}}\, dx$ Rj. $2\sqrt{x^3-2} + C$

22.ˇ $\int \operatorname{tg} x\, dx$ Rj. $-\ln|\cos x| + C$

23.ˇ $\int \frac{\sin x}{\sqrt{5\cos x-2}}\, dx$ Rj. $-\frac{2}{5}\sqrt{5\cos x-2} + C$

24.
$$\int e^{\cos x}\cdot\sin x\,dx = \left|\begin{array}{l}\cos x = t\\ -\sin x\,dx = dt\\ \sin x\,dx = -dt\end{array}\right| = \int e^{t}\cdot(-dt) = -\int e^{t}dt =$$
$$= -e^{t}+c = -e^{\cos x}+c$$

25.
$$\int\frac{dx}{x\sqrt[5]{\ln x}} = \left|\begin{array}{l}\ln x = t^{5}\\ \frac{1}{x}dx = 5t^{4}dt\\ t = \sqrt[5]{\ln x}\end{array}\right| = \int\frac{5t^{4}dt}{t} = 5\int t^{3}dt = 5\cdot\frac{t^{4}}{4}+c$$
$$= \frac{5}{4}\sqrt[5]{\ln^{4}x}+c$$

26.
$$\int\frac{x^{3}dx}{x^{8}-2} = \int\frac{x^{3}\,dx}{(x^{4})^{2}-2} = \left|\begin{array}{l}x^{4}=t\\ 4x^{3}dx=dt\\ x^{3}dx=\frac{1}{4}dt\end{array}\right| = \frac{1}{4}\int\frac{dt}{t^{2}-2} = \frac{1}{4}\cdot\frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+c$$
$$= \frac{1}{8\sqrt{2}}\ln\left|\frac{x^{4}-\sqrt{2}}{x^{4}+\sqrt{2}}\right|+c$$

27.
$$\int\frac{\sqrt[3]{\mathrm{tg}\,x}}{\cos^{2}x}dx = \left|\begin{array}{l}\mathrm{tg}\,x = t^{3}\\ \frac{1}{\cos^{2}x}dx = 3t^{2}dt\end{array}\right| = \int\sqrt[3]{t^{3}}\cdot 3t^{2}dt = 3\int t^{3}dt =$$
$$= 3\frac{t^{4}}{4}+c = \frac{3}{4}\sqrt[3]{\mathrm{tg}^{4}x}+c$$

28.
$$\int x(1-x)^{10}dx = \left|\begin{array}{l}1-x=t\\ -dx=dt\\ dx=-dt\\ x=1-t\end{array}\right| = \int(1-t)\,t^{10}\cdot(-dt) = -\int(t^{10}-t^{11})dt$$
$$= -\frac{t^{11}}{11}+\frac{t^{12}}{12}+c = \frac{(1-x)^{12}}{12}-\frac{(1-x)^{11}}{11}+c$$

29.
$$\int\frac{x^{2}-1\;\;/:x^{2}}{x^{4}+1\;\;/:x^{2}}dx = \int\frac{1-\frac{1}{x^{2}}}{x^{2}+\frac{1}{x^{2}}}dx = \left|\begin{array}{l}\text{treba da stavimo zamjenu takvu da dobijemo } \left(1-\frac{1}{x^{2}}\right)dx = dt\\ t = x+\frac{1}{x} \Rightarrow \left(1-\frac{1}{x^{2}}\right)dx = dt\\ \left(x+\frac{1}{x}\right)^{2} = t^{2} \Rightarrow x^{2}+2x\cdot\frac{1}{x}+\frac{1}{x^{2}} = t^{2}\\ x^{2}+\frac{1}{x^{2}} = t^{2}-2\end{array}\right|$$
$$= \int\frac{dt}{t^{2}-2} = \frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+c = \frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|+c$$

30.
$$\int\sqrt{\frac{\arcsin x}{1-x^{2}}}\,dx = \int\frac{\sqrt{\arcsin x}}{\sqrt{1-x^{2}}}dx = \left|\begin{array}{l}\arcsin x = t^{2}\\ \frac{dx}{\sqrt{1-x^{2}}} = 2t\,dt\\ t = \sqrt{\arcsin x}\end{array}\right| =$$
$$= \int\sqrt{t^{2}}\,2t\,dt = 2\int t^{2}dt = 2\cdot\frac{t^{3}}{3}+c = \frac{2}{3}\sqrt{\arcsin^{3}x}+c$$

31.
$$\int(x+4)\sqrt[5]{2x-1}\,dx = \left|\begin{array}{ll}2x-1 = t^{5} & 2x = t^{5}+1\\ 2\,dx = 5t^{4}dt & x = \frac{t^{5}+1}{2}\\ dx = \frac{5}{2}t^{4}dt & t = \sqrt[5]{2x-1}\end{array}\right| = \int\left(\frac{t^{5}+1}{2}+4\right)\sqrt[5]{t^{5}}\cdot\frac{5}{2}t^{4}dt$$
$$= \frac{5}{2}\int\frac{t^{5}+1+8}{2}\,t^{5}dt = \frac{5}{4}\int(t^{5}+9)t^{5}dt = \frac{5}{4}\int(t^{10}+9t^{5})dt =$$
$$= \frac{5}{4}\cdot\frac{t^{11}}{11}+\frac{5}{4}\cdot\overset{3}{\cancel{9}}\cdot\frac{t^{6}}{\underset{2}{\cancel{6}}}+c = \frac{5}{44}\sqrt[5]{(2x-1)^{11}}+\frac{15}{8}\sqrt[5]{(2x-1)^{6}}+c$$

32. ISPITNI
$$\int\frac{\ln x\,dx}{x\sqrt{1+\ln x}} = \left|\begin{array}{l}\ln x = t\\ \frac{1}{x}dx = dt\end{array}\right| = \int\frac{t+1-1}{\sqrt{1+t}}dt = \int\frac{\overset{\cdot\sqrt{1+t}}{t+1}}{\underset{\cdot\sqrt{1+t}}{\sqrt{1+t}}}dt-\int\frac{dt}{\sqrt{1+t}}$$
$$= \int(1+t)^{\frac{1}{2}}dt-\int(1+t)^{-\frac{1}{2}}dt = \frac{(1+t)^{\frac{3}{2}}}{\frac{3}{2}}-\frac{(1+t)^{\frac{1}{2}}}{\frac{1}{2}}+c = \frac{2}{3}(1+t)^{\frac{3}{2}}-2(1+t)^{\frac{1}{2}}+c$$
$$= \frac{2}{3}(1+t)^{\frac{1}{2}}\cdot((1+t)-3)+c = \frac{2}{3}\sqrt{1+\ln x}\,(\ln x-2)+c$$

33. ✓ ISPITNI
$$\int\frac{e^{3x}(10-2e^{3x})}{2e^{6x}-10e^{3x}+12}dx$$
Rj. $-\frac{1}{6}\ln|e^{6x}-5e^{3x}+6|+\frac{5}{6}\ln\left|\frac{e^{3x}-3}{e^{3x}-2}\right|+c$

34. ✓
$$\int\frac{1}{x^{2}}\sin\frac{1}{x}\,dx$$

35. ✓
$$\int x(2x+3)^{7}dx$$

36. ✓
$$\int\frac{\arcsin x}{\sqrt{1-x^{2}}}dx$$

37. ✓
$$\int\frac{x^{2}+1}{\sqrt{x^{6}-7x^{4}+x^{2}}}dx$$

(#) Izračunati integral $\int\sqrt{\frac{x-2}{x+2}}\,dx$.

Rj.

$$\int\sqrt{\frac{x-2}{x+2}}\,dx=\int\frac{\sqrt{x-2}}{\sqrt{x+2}}\,dx=\int\frac{\sqrt{x-2}\cdot\sqrt{x-2}}{\sqrt{x+2}\cdot\sqrt{x-2}}\,dx=\int\frac{x-2}{\sqrt{x^2-4}}\,dx$$

$$=\int\frac{x}{\sqrt{x^2-4}}\,dx-2\int\frac{dx}{\sqrt{x^2-4}}$$

$$\int\frac{x}{\sqrt{x^2-4}}\,dx=\left|\begin{array}{l}x^2-4=t\\2x\,dx=dt\\x\,dx=\frac{1}{2}\,dt\end{array}\right|=\frac{1}{2}\int\frac{dt}{\sqrt{t}}=\frac{1}{2}\int t^{-\frac{1}{2}}\,dt=\frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}}+c$$

$$=\sqrt{t}+c=\sqrt{x^2-4}+c$$

$$\int\frac{dx}{\sqrt{x^2-4}}=\left|\begin{array}{l}x=2s\\dx=2\,ds\\s=\frac{1}{2}x\end{array}\right|=\int\frac{2\,ds}{\sqrt{4s^2-4}}=\frac{2}{\sqrt{4}}\int\frac{ds}{\sqrt{s^2-1}}=\ln|s+\sqrt{s^2-1}|+c_1$$

$$=\ln\left|\frac{1}{2}x+\sqrt{\frac{1}{4}x^2-1}\right|+c_1=\ln\left|\frac{1}{2}x+\frac{1}{2}\sqrt{x^2-4}\right|+c_1$$

$$=\ln\frac{1}{2}+\ln|x+\sqrt{x^2-4}|+c_1=\ln|x+\sqrt{x^2-4}|+c$$

$$\int\sqrt{\frac{x-2}{x+2}}\,dx=\sqrt{x^2-4}-2\ln|x+\sqrt{x^2-4}|+c$$

(#) Izračunati integral $\int x^3\sqrt{1+a^2x^2}\,dx$, $(a>0)$.

Rj.

$$\int x^3\sqrt{1+a^2x^2}\,dx=\int x^2\cdot x\cdot\sqrt{1+a^2x^2}\,dx=\left|\begin{array}{l}1+a^2x^2=t^2\\a^2\cdot2x\,dx=2t\,dt\\x\,dx=\frac{1}{a^2}\,t\,dt\\a^2x^2=t^2-1\\x^2=\frac{1}{a^2}(t^2-1)\end{array}\right|$$

$$=\int\frac{1}{a^2}(t^2-1)\cdot\frac{1}{a^2}\,t\cdot t\,dt=$$

$$=\frac{1}{a^4}\int(t^4-t^2)\,dt=\frac{1}{a^4}\cdot\frac{1}{5}t^5-\frac{1}{a^4}\cdot\frac{1}{3}t^3=$$

$$=\frac{1}{5a^4}\sqrt{(1+a^2x^2)^5}-\frac{1}{3a^4}\sqrt{(1+a^2x^2)^3}+c$$

(#) Izračunati integral $\int\frac{dx}{\sqrt[4]{x^3}(1+\sqrt[6]{x})}$.

Rj.

$$\int\frac{dx}{\sqrt[4]{x^3}(1+\sqrt[6]{x})}=\left|\begin{array}{ll}x=t^{12} & t=\sqrt[12]{x}\\dx=12t^{11}dt & \\\sqrt[4]{x^3}=\sqrt[4]{t^{36}}=t^9 & \\\sqrt[6]{x}=\sqrt[6]{t^{12}}=t^2 & \end{array}\right|=\int\frac{12t^{11}\,dt}{t^9(1+t^2)}=$$

$$=12\int\frac{t^{2}\,{}^{+1-1}}{1+t^2}\,dt=12\int\frac{t^2+1}{t^2+1}\,dt-12\int\frac{dt}{1+t^2}=$$

$$=12t-12\,\mathrm{arc\,tg}\,t+c=12\sqrt[12]{x}-12\,\mathrm{arc\,tg}\sqrt[12]{x}+c$$

4. Metoda parcijalne integracije

Prema formuli diferenciranja znamo da je

$$d(uv) = u\,dv + v\,du$$

a znamo i da je $\int d(uv) = uv$. Prema tome nije teško vidjeti da vrijedi

$$\int u\,dv = uv - \int v\,du \qquad \dots(*)$$

Formulu $(*)$ nazivamo formula parcijalne integracije.

Posmatrajmo neki dati integral $\int f(x)g(x)\,dx$. Metodom parcijalne integracije biramo f-ju $u=f(x)$ i $dv=g(x)\,dx$ tako da se integral $\int v\,du$ može jednostavno riješiti.

(#) Odrediti integrale

a) $\int x\cos x\,dx$ b) $\int \frac{\ln x}{x^3}\,dx$ c) $\int x\,\mathrm{arctg}\,x\,dx$

d) $\int \arcsin x\,dx$ e) $\int x^2 e^{3x}\,dx$ f) $\int e^{-x}\cos\frac{x}{2}\,dx$

Rj.

a) $$\int x\cos x\,dx = \left|\begin{array}{ll} u=x & dv=\cos x\,dx \\ du=dx & v=\int\cos x\,dx=\sin x \end{array}\right| =$$

$$= x\sin x - \int \sin x\,dx = x\sin x + \cos x + C$$

b) $$\int \frac{\ln x}{x^3}\,dx = \left|\begin{array}{ll} u=\ln x & dv=\frac{dx}{x^3} \\ du=\frac{1}{x}\,dx & v=\int\frac{dx}{x^3}=\frac{x^{-2}}{-2}=-\frac{1}{2x^2} \end{array}\right| =$$

$$= -\frac{\ln x}{2x^2} - \int \frac{(-1)}{2x^2}\cdot\frac{1}{x}\,dx = -\frac{\ln x}{2x^2} + \frac{1}{2}\int\frac{dx}{x^3} =$$

$$= -\frac{1}{2x^2}\ln x + \frac{1}{2}\cdot\frac{x^{-2}}{-2} + c = -\frac{1}{2x^2}\ln x - \frac{1}{4x^2} + C$$

$$= C - \frac{2\ln x + 1}{4x^2} + c$$

c) $$\int x\,\mathrm{arctg}\,x\,dx = \left|\begin{array}{ll} u=\mathrm{arc\,tg}\,x & dv=x\,dx \\ du=\frac{dx}{1+x^2} & v=\int x\,dx=\frac{x^2}{2} \end{array}\right| =$$

$$= \frac{1}{2}x^2\,\mathrm{arctg}\,x - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx \qquad \dots(1)$$

Izračunajmo posebno $\int\frac{x^2}{1+x^2}\,dx$

$$\int \frac{x^2}{1+x^2}\,dx = \int \frac{x^2+1-1}{1+x^2}\,dx = \int \frac{1+x^2-1}{1+x^2}\,dx =$$

$$= \int \left(1-\frac{1}{1+x^2}\right)dx = x - \operatorname{arctg} x + c_1$$

Sad, prema (1) imamo

$$I = \frac{1}{2}x^2 \operatorname{arctg} x - \frac{1}{2}x + \frac{1}{2}\operatorname{arctg} x + c =$$

$$= c - \frac{1}{2}x + \frac{x^2+1}{2}\operatorname{arctg} x$$

d) $$\int \arcsin x\,dx = \left| \begin{matrix} u=\arcsin x & dv = dx \\ du = \frac{dx}{\sqrt{1-x^2}} & v=\int dx = x \end{matrix} \right| =$$

$$= x\arcsin x - \int \frac{x\,dx}{\sqrt{1-x^2}} = \left| \begin{matrix} d(1-x^2) = -2x\,dx \\ x\,dx = -\frac{1}{2}\,d(1-x^2) \end{matrix} \right| =$$

$$= x\arcsin x + \frac{1}{2}\int \frac{d(1-x^2)}{\sqrt{1-x^2}} = x\arcsin x + \frac{1}{2}\int (1-x^2)^{-\frac{1}{2}}\,d(1-x^2)$$

$$= x\arcsin x + \frac{1}{2}\,\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}} + C = x\arcsin x + \sqrt{1-x^2} + C$$

e) $$\int x^2 e^{3x}\,dx = \left| \begin{matrix} u=x^2 & dv = e^{3x}dx \\ du=2x\,dx & v=\int e^{3x}dx = \frac{1}{3}\int e^{3x}d(3x) = \frac{1}{3}e^{3x} \end{matrix} \right| =$$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\int x\, e^{3x}\,dx \qquad \dots (1)$$

$$\int x\, e^{3x}\,dx = \left| \begin{matrix} u=x & dv=e^{3x}dx \\ du=dx & v=\frac{1}{3}e^{3x} \end{matrix} \right| = \frac{1}{3}x e^{3x} - \frac{1}{3}\int e^{3x}dx =$$

$$= \frac{1}{3}x e^{3x} - \frac{1}{3}\cdot\frac{1}{3}e^{3x} + c_1 = \frac{1}{3}x e^{3x} - \frac{1}{9}e^{3x} + c_1$$

Prema (1) imamo

$$\int x^2 e^{3x}dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{3}\left(\frac{1}{3}x e^{3x} - \frac{1}{9}e^{2x} + c_1\right) =$$

$$= \frac{1}{3}x^2 e^{3x} - \frac{2}{9}x e^{3x} + \frac{2}{27}e^{3x} + C$$

$$= \frac{1}{27}e^{3x}\left(9x^2 - 6x + 2\right) + C$$

f) $$I = \int e^{-x}\cos\frac{x}{2}\,dx = \left| \begin{matrix} u=e^{-x} & dv = \cos\frac{x}{2}\,dx \\ du = e^{-x}\cdot(-1) = -e^{-x}dx & v=\int \cos\frac{x}{2}dx = 2\int \cos\frac{x}{2}\,d\left(\frac{x}{2}\right) \\ & = 2\sin\frac{x}{2} \end{matrix} \right|$$

$$= 2e^{-x}\sin\frac{x}{2} + 2\int e^{-x}\sin\frac{x}{2}\,dx = \left| \begin{matrix} u=e^{-x} & dv=\sin\frac{x}{2}\,dx \\ du=-e^{-x}dx & v=(-2)\cos\frac{x}{2} \end{matrix} \right| =$$

$$= 2e^{-x}\sin\frac{x}{2} + 2\left((-2)\,e^{-x}\cos\frac{x}{2} - 2\int e^{-x}\; s\frac{x}{2}\,dx\right)$$

$$= 2e^{-x}\sin\frac{x}{2} - 4e^{-x}\cos\frac{x}{2} - 4\int e^{-x}\cos\frac{x}{2}\,dx$$

Tj.

$$I = 2e^{-x}\sin\frac{x}{2} - 4e^{-x}\cos\frac{x}{2} - 4I$$

$$5I = 2e^{-x}\sin\frac{x}{2} - 4e^{-x}\cos\frac{x}{2}$$

$$I = \frac{2}{5}e^{-x}\left(\sin\frac{x}{2} - 2\cos\frac{x}{2}\right) + C$$

Zadaci za vježbu

① $\int x \sin x \, dx$ ② $\int x^2 \ln x \, dx$ ③ $\int \ln(x^n) \, dx$

④ $\int (x^2+1) e^{-2x} \, dx$ ⑤ $\int \frac{x}{\cos^2 x} \, dx$ ⑥ $\int x \ln(x-1) \, dx$

⑦ $\int \operatorname{arc\,ctg} t \, dt$ ⑧ $\int \ln(1+x^2) \, dx$ ⑨ $\int e^{ax} \sin bx \, dx$

⑩* $\int \frac{\arcsin x}{x^2} \, dx$ ⑪* $\int \frac{\ln x \, dx}{(x+1)^2}$

⑫* $\int \operatorname{arc\,tg} \sqrt{2x-1} \, dx$

Rješenja

1. $\sin x - x \cos x$ 2. $\frac{x^3}{9}(3\ln x - 1)$ 3. $nx(\ln x - 1)$

4. $-\frac{2x^2+2x+3}{4e^{2x}}$ 5. $x \operatorname{tg} x + \ln|\cos x|$ 6. $\frac{x^2-1}{2} \ln|x-1| - \frac{x^2}{4} - \frac{x}{2}$ 7. $t \operatorname{arc\,ctg} t + \frac{1}{2} \ln(1+t^2)$

8. $x \ln(x^2+1) - 2x + 2 \operatorname{arc\,tg} x$ 9. $\frac{e^{ax}(a \sin bx - b \cos bx)}{a^2+b^2}$

10. $\ln \frac{1-\sqrt{1-x^2}}{x} - \frac{1}{x} \arcsin x$ 11. $\frac{x \ln x}{x+1} - \ln|x+1|$

12. $x \operatorname{arc\,tg} \sqrt{2x-1} - \frac{1}{2}\sqrt{2x-1}$

Izabrani **Zadaci za vježbu** sa rješenjima (iz lekcije Metoda parcijalne integracije)

①
$$\int x e^x \, dx = \left| \begin{matrix} u = x & dv = e^x dx \\ du = dx & v = \int e^x dx = e^x \end{matrix} \right| = x e^x - \int e^x dx = x e^x - e^x + C$$

Da smo uzeli smjene $u = e^x$, $dv = x\,dx$ dobili bi komplikovan izraz.

②
$$\int x^2 \sin 3x \, dx = \left| \begin{matrix} u = x^2 & dv = \sin 3x \, dx \\ du = 2x \, dx & v = \int \sin 3x \, dx = -\frac{1}{3} \cos 3x \end{matrix} \right| \overset{(*)}{=}$$

$$\int \sin 3x \, dx = \left| \begin{matrix} 3x = t \\ 3\,dx = dt \\ d = \frac{1}{3} dt \end{matrix} \right| = \frac{1}{3} \int \sin t \, dt = -\frac{1}{3} \cos t + c = -\frac{1}{3} \cos 3x + c$$

$$\overset{(*)}{=} -\frac{1}{3} x^2 \cos 3x + \frac{2}{3} \int x \cos 3x \, dx = -\frac{1}{3} x^2 \cos 3x + \frac{2}{3} I_1$$

$$I_1 = \int x \cos 3x \, dx = \left| \begin{matrix} u = x & dv = \cos 3x \, dx \\ du = dx & v = \frac{1}{3} \sin 3x \end{matrix} \right| = \frac{1}{3} x \sin 3x - \frac{1}{3} \int \sin 3x \, dx$$

$$= \frac{1}{3} x \sin 3x - \frac{1}{3} \cdot \left(-\frac{1}{3}\right) \cos 3x + C_1 = \frac{1}{3} x \sin 3x + \frac{1}{9} \cos 3x + C_1$$

$$\int x^2 \sin 3x \, dx = -\frac{1}{3} x^2 \cos 3x + \frac{2}{9} x \sin 3x + \frac{2}{27} \cos 3x + C$$

③
$$\int x^3 \ln x \, dx = \left| \begin{matrix} u = \ln x & dv = x^3 dx \\ du = \frac{1}{x} dx & v = \frac{x^4}{4} \end{matrix} \right| = \frac{1}{4} x^4 \ln x - \frac{1}{4} \int x^3 dx$$

$$= \frac{1}{4} x^4 \ln x - \frac{1}{4} \cdot \frac{x^4}{4} + C = \frac{1}{4} x^4 \ln x - \frac{1}{16} x^4 + C$$

④
$$\int \arcsin x \, dx = \left| \begin{matrix} u = \arcsin x & dv = dx \\ du = \frac{1}{\sqrt{1-x^2}} dx & v = x \end{matrix} \right| = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$= x \arcsin x - I_1$$

$$I_1=\int\frac{x}{\sqrt{1-x^2}}\,dx=\left|\begin{matrix}1-x^2=t^2\\ -2x\,dx=2t\,dt\\ x\,dx=-t\,dt\\ t=\sqrt{1-x^2}\end{matrix}\right|=\int\frac{-t\,dt}{t}=-\int dt=-t+C_1=$$

$$=-\sqrt{1-x^2}+C_1$$

$$\int\arcsin x\,dx=x\arcsin x+\sqrt{1-x^2}+C$$

5. ISPITNI ZADATAK

$$\int\sin(\ln x)\,dx=\left|\begin{matrix}u=\sin(\ln x) & dv=dx\\ du=\cos(\ln x)\cdot\frac{1}{x}\,dx & v=x\end{matrix}\right|=x\sin(\ln x)-\int\cos(\ln x)\,dx$$

$$\int\cos(\ln x)\,dx=\left|\begin{matrix}u=\cos(\ln x) & dv=dx\\ du=-\sin(\ln x)\cdot\frac{1}{x}\,dx & v=x\end{matrix}\right|=x\cos(\ln x)+\int\sin(\ln x)\,dx$$

$$\underline{\underline{\int\sin(\ln x)\,dx}}=x\sin(\ln x)-x\cos(\ln x)\underline{\underline{-\int\sin(\ln(x)\,dx}}$$

$$2\int\sin(\ln x)\,dx=x\sin(\ln x)-x\cos(\ln x)\qquad /:2$$

$$\int\sin(\ln x)\,dx=\frac{x}{2}\sin(\ln x)-\frac{x}{2}\cos(\ln x)+C$$

6.

$$\int x\,\mathrm{arctg}\,x\,dx=\left|\begin{matrix}u=\mathrm{arctg}\,x & dv=x\,dx\\ du=\frac{1}{1+x^2}\,dx & v=\frac{x^2}{2}\end{matrix}\right|=\frac{1}{2}x^2\,\mathrm{arctg}\,x-\frac{1}{2}\int\frac{x^2}{1+x^2}\,dx$$

$$\int\frac{x^2}{x^2+1}\,dx=\int\frac{x^2+1-1}{x^2+1}\,dx=\int dx-\int\frac{1}{x^2+1}\,dx=x-\mathrm{arctg}\,x+C_1$$

$$\int x\,\mathrm{arctg}\,x\,dx=\frac{1}{2}x^2\,\mathrm{arctg}\,x-\frac{1}{2}x+\frac{1}{2}\,\mathrm{arctg}\,x+C$$

7.^{V} $\int x^2e^{-2x}\,dx$ Rj: $-\frac{x^2}{2}e^{-2x}-\frac{x}{2}e^{-2x}-\frac{1}{4}e^{-2x}+C$

8.^{V} $\int\frac{x^2}{(x^2+1)^2}\,dx$ Rj: $-\frac{x}{2(x^2+1)}+\frac{1}{2}\,\mathrm{arctg}\,x+C$

uputa: $\int x\cdot\frac{x}{(x^2+1)^2}\,dx=\left|\begin{matrix}u=x & dv=\frac{x\,dx}{(x^2+1)^2}\end{matrix}\right|=\ldots$

9.

$$\int x^3\ln(2x+1)\,dx=\left|\begin{matrix}u=\ln(2x+1) & dv=x^3\,dx\\ du=\frac{1}{2x+1}\cdot 2\,dx & v=\frac{x^4}{4}\end{matrix}\right|=$$

$$=\frac{1}{4}x^4\ln(2x+1)-\frac{1}{4}\cdot 2\int\frac{x^4}{2x+1}\,dx=\frac{1}{4}x^4\ln(2x+1)-\frac{1}{2}I_1$$

$$\begin{array}{l}x^4:(2x+1)=\frac{1}{2}x^3-\frac{1}{4}x^2+\frac{1}{8}x-\frac{1}{16}\\ \underline{-\;x^4+\frac{1}{2}x^3}\\ \quad-\frac{1}{2}x^3\\ \underline{-\;-\frac{1}{2}x^3-\frac{1}{4}x^2}\\ \qquad\frac{1}{4}x^2\\ \qquad\underline{-\;\frac{1}{4}x^2+\frac{1}{8}x}\\ \qquad\qquad-\frac{1}{8}x\\ \qquad\qquad\underline{-\;-\frac{1}{8}x-\frac{1}{16}}\\ \qquad\qquad\text{ostatak }\frac{1}{16}\end{array}$$

ostatak $\frac{1}{16}$

$$x^4=\left(\frac{1}{2}x^3-\frac{1}{4}x^2+\frac{1}{8}x-\frac{1}{16}\right)(2x+1)+\frac{1}{16}$$

$$I_1=\int\left(\frac{1}{2}x^3-\frac{1}{4}x^2+\frac{1}{8}x-\frac{1}{16}+\frac{\frac{1}{16}}{2x+1}\right)dx=$$

$$=\frac{1}{2}\cdot\frac{x^4}{4}-\frac{1}{4}\cdot\frac{x^3}{3}+\frac{1}{8}\cdot\frac{x^2}{2}-\frac{1}{16}x+\frac{1}{16}\cdot\frac{1}{2}\ln|2x+1|+C_1$$

$$\int x^3\ln(2x+1)\,dx=\frac{1}{4}x^4\ln(2x+1)-\frac{1}{16}x^4+\frac{1}{24}x^3-\frac{1}{32}x^2+\frac{1}{32}x-\frac{1}{64}\ln|2x+1|+C$$

10. ISPITNI ZADATAK

$$I=\int\sqrt{x}\,\ln^2x\,dx=\left|\begin{matrix}u=\ln^2x & dv=\sqrt{x}\,dx\\ du=2\ln x\cdot\frac{1}{x}\,dx & v=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}=\frac{2}{3}\sqrt{x^3}\end{matrix}\right|=$$

$$=\frac{2}{3}\sqrt{x^3}\ln^2x-2\cdot\frac{2}{3}\int\frac{\sqrt{x^3}}{x}\ln x\,dx=\frac{2}{3}x\sqrt{x}\ln^2x-\frac{4}{3}\int\sqrt{x}\ln x\,dx$$

$$\int\sqrt{x}\ln x\,dx=\left|\begin{matrix}u=\ln x & dv=\sqrt{x}\,dx\\ du=\frac{1}{x}\,dx & v=\frac{2}{3}\sqrt{x^3}\end{matrix}\right|=\frac{2}{3}x\sqrt{x}\ln x-\frac{2}{3}\int\frac{x\sqrt{x}}{x}\,dx=$$

$$=\frac{2}{3}x\sqrt{x}\ln x-\frac{2}{3}\cdot\frac{2}{3}\sqrt{x^3}+C=\frac{2}{3}x\sqrt{x}\ln x-\frac{4}{9}x\sqrt{x}+C$$

$$I=\frac{2}{3}x\sqrt{x}\ln^2x-\frac{8}{9}x\sqrt{x}\ln x+\frac{16}{27}x\sqrt{x}+C$$

⑪ ISPITNI ZADATAK

$$I=\int e^{3x}\cos 4x\,dx=\left|\begin{matrix} u=e^{3x} & dv=\cos 4x\,dx \\ du=3e^{3x}dx & v=\frac{1}{4}\sin 4x \end{matrix}\right|=$$

$$=\frac{1}{4}e^{3x}\sin 4x-\frac{3}{4}\int e^{3x}\sin 4x\,dx=\frac{1}{4}e^{3x}\sin 4x-\frac{3}{4}I_1$$

$$I_1=\int e^{3x}\sin 4x\,dx=\left|\begin{matrix} u=e^{3x} & dv=\sin 4x\,dx \\ du=3e^{3x}dx & v=-\frac{1}{4}\cos 4x \end{matrix}\right|=$$

$$=-\frac{1}{4}e^{3x}\cos 4x+\frac{3}{4}\int e^{3x}\cos 4x\,dx=-\frac{1}{4}e^{3x}\cos 4x+\frac{3}{4}I$$

$$I=\frac{1}{4}e^{2x}\sin 4x+\frac{3}{16}e^{3x}\cos 4x-\frac{9}{16}I \qquad /\cdot 16$$

$$16I=4e^{2x}\sin 4x+3e^{3x}\cos 4x-9I$$

$$I=\frac{4e^{3x}\sin 4x+3e^{3x}\cos 4x}{25}+C$$

⑫ ISPITNI ZADATAK

$$I=\int x^2e^{3x}dx=\left|\begin{matrix} u=x^2 & dv=e^{3x}dx \\ du=2x\,dx & v=\frac{1}{3}e^{3x} \end{matrix}\right|=\frac{1}{3}x^2e^{3x}-\frac{2}{3}\int xe^{3x}dx$$

$$\int xe^{3x}dx=\left|\begin{matrix} u=x & dv=e^{3x}dx \\ du=dx & v=\frac{1}{3}e^{3x} \end{matrix}\right|=\frac{1}{3}xe^{3x}-\frac{1}{3}\int e^{3x}dx=\frac{1}{3}xe^{3x}-\frac{1}{9}e^{3x}+C$$

$$I=\frac{1}{3}x^2e^{3x}-\frac{2}{9}xe^{3x}+\frac{2}{27}e^{3x}+C$$

⑬ ✓ $I=\int\frac{\arcsin\frac{x}{2}}{\sqrt{2-x}}dx$ Uputa: $u=\arcsin\frac{x}{2}$, $dv=\frac{dx}{\sqrt{2-x}}$ Rj. $-2\sqrt{2-x}\cdot\arcsin\frac{x}{2}+$ $+4\sqrt{2+x}+C$

⑭ ✓ $I=\int e^x\sin x\,dx$ Uputa: $u=e^x$ Rj. $\frac{e^x(\sin x-\cos x)}{2}+C$

⑮ ✓ $\int x^5\ln x\,dx$

⑯ ✓ $\int\frac{x^2\,dx}{\cos^2 x}$

⑰ ✓ $\int\frac{\ln(x^2+1)}{x^3}dx$

⑱ ✓ $\int(\arcsin x)^2dx$

(#) Izračunati integral $I=\int x^3e^{\frac{x}{2}}dx$.

Rj.

$$\int e^{\frac{x}{2}}dx=\left|\begin{matrix} \frac{x}{2}=t \\ \frac{1}{2}dx=dt \\ dx=2\,dt \end{matrix}\right|=2\int e^t dt=2e^t+c=2e^{\frac{x}{2}}+c$$

$$\int x^3e^{\frac{x}{2}}dx=\left|\begin{matrix} u=x^3 & dv=e^{\frac{x}{2}}dx \\ du=3x^2dx & v=2e^{\frac{x}{2}} \end{matrix}\right|=2x^3e^{\frac{x}{2}}-6\int x^2e^{\frac{x}{2}}dx$$

$$\int x^2e^{\frac{x}{2}}dx=\left|\begin{matrix} u=x^2 & dv=e^{\frac{x}{2}}dx \\ du=2x\,dx & v=2e^{\frac{x}{2}} \end{matrix}\right|=2x^2e^{\frac{x}{2}}-4\int xe^{\frac{x}{2}}dx$$

$$\int xe^{\frac{x}{2}}dx=\left|\begin{matrix} u=x & dv=e^{\frac{x}{2}}dx \\ du=dx & v=2e^{\frac{x}{2}} \end{matrix}\right|=2xe^{\frac{x}{2}}-2\int e^{\frac{x}{2}}dx=2xe^{\frac{x}{2}}-4e^{\frac{x}{2}}+c$$

$$I=2x^3e^{\frac{x}{2}}-6\left[2x^2e^{\frac{x}{2}}-4\left(2xe^{\frac{x}{2}}-4e^{\frac{x}{2}}\right)\right]+c=$$

$$=2x^3e^{\frac{x}{2}}-6\left(2x^2e^{\frac{x}{2}}-8xe^{\frac{x}{2}}+16e^{\frac{x}{2}}\right)+c$$

$$=2x^3e^{\frac{x}{2}}-12x^2e^{\frac{x}{2}}+48xe^{\frac{x}{2}}-96e^{\frac{x}{2}}+c$$

$$I=2e^{\frac{x}{2}}\left(x^3-6x^2+24x-48\right)+c$$

Ⓘ# Izračunati: $\int\sqrt{5-x^2}\,dx$.

Rj. $$\int\sqrt{5-x^2}\,dx = \left|\begin{array}{l} x=\sqrt{5}\,t \\ dx=\sqrt{5}\,dt \\ t=x/\sqrt{5}\end{array}\right| = \int\sqrt{5-5t^2}\,\sqrt{5}\,dt = 5\int\sqrt{1-t^2}\,dt$$

$$\int\sqrt{1-t^2}\,dt = \left|\begin{array}{ll} u=\sqrt{1-t^2} & dv=dt \\ du=\frac{-2t}{2\sqrt{1-t^2}}=\frac{-t}{\sqrt{1-t^2}}dt & v=t\end{array}\right| = t\sqrt{1-t^2}+\int\frac{t^2}{\sqrt{1-t^2}}dt$$

$$\int\frac{t^2}{\sqrt{1-t^2}} = \int\frac{t^2-1+1}{\sqrt{1-t^2}}dt = -\int\frac{1-t^2}{\sqrt{1-t^2}}dt+\int\frac{dt}{\sqrt{1-t^2}} =$$

$$= -\int\sqrt{1-t^2}\,dt + \arcsin t$$

$$\int\sqrt{1-t^2}\,dt = t\sqrt{1-t^2} - \int\sqrt{1-t^2} + \arcsin t$$

$$2\int\sqrt{1-t^2}\,dt = t\sqrt{1-t^2} + \arcsin t$$

$$\int\sqrt{1-t^2} = \frac{1}{2}\left(t\sqrt{1-t^2}+\arcsin t\right) + C$$

$$\int\sqrt{5-x^2}\,dx = \frac{5}{2}\left(\frac{x}{\sqrt{5}}\sqrt{1-\frac{x^2}{5}}+\arcsin\frac{x}{\sqrt{5}}\right)+C =$$

$$= \frac{5}{2}\left(\frac{x\sqrt{5}}{5}\sqrt{\frac{5-x^2}{5}}+\arcsin\frac{x\sqrt{5}}{5}\right)+C =$$

$$= \frac{x}{2}\sqrt{5-x^2}+\frac{5}{2}\arcsin\frac{x\sqrt{5}}{5} + C$$

Ⓘ# Izračunati integral $\int x\sqrt{1-x^4}\,dx$

Rj. $$\int x\sqrt{1-x^4}\,dx = \int x\sqrt{(1-x^2)(1+x^2)}\,dx = \left|\begin{array}{l} x^2=t \\ 2x\,dx=dt \\ x\,dx=\frac{1}{2}dt\end{array}\right| = \frac{1}{2}\int\sqrt{(1-t)(1+t)}\,dt =$$

$$= \frac{1}{2}\int\sqrt{1-t^2}\,dt = \frac{1}{2}\int\frac{1-t^2}{\sqrt{1-t^2}}dt = \frac{1}{2}\left[\int\frac{dt}{\sqrt{1-t^2}} - \int\frac{t^2}{\sqrt{1-t^2}}dt\right]$$

$$\int\frac{t^2}{\sqrt{1-t^2}}dt = \left|\begin{array}{ll} u=t & dv=\frac{t}{\sqrt{1-t^2}}dt \\ du=dt & v=\int\frac{t}{\sqrt{1-t^2}}dt = \left|\begin{array}{l} 1-t^2=s^2 \\ -2t\,dt=2s\,ds \\ t\,dt=-s\,ds\end{array}\right| = -\int\frac{s\,ds}{s} = -\int ds \\ & = -s = -\sqrt{1-t^2}\end{array}\right| =$$

$$= -t\sqrt{1-t^2} + \int\sqrt{1-t^2}\,dt$$ Sad imamo:

$$\underline{\underline{\frac{1}{2}\int\sqrt{1-t^2}\,dt}} = \frac{1}{2}\arcsin t + \frac{1}{2}t\sqrt{1-t^2} \underline{\underline{- \frac{1}{2}\int\sqrt{1-t^2}\,dt}}$$

$$\int\sqrt{1-t^2}\,dt = \frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\arcsin t$$

vratimo smjene $$\frac{1}{2}\int\sqrt{1-t^2} = \frac{1}{2}\left(\frac{1}{2}t\sqrt{1-t^2}+\frac{1}{2}\arcsin t\right)$$

$$\int x\sqrt{1-x^4}\,dx = \frac{1}{4}x^2\sqrt{1-x^4} + \frac{1}{4}\arcsin x^2 + C$$

5. Integracija kvadratnog trinoma

U ovoj lekciji rješavamo integrale sljedećih oblika

$$\int\frac{Ax+B}{ax^2+bx+c}\,dx;\qquad \int\frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx;\qquad \int\sqrt{ax^2+bx+c}\,dx.$$

Kvadratni trinom ax^2+bx+c, koji se pojavljuje u podintegralnoj f-ji uvijek svodimo na "pogodniji" oblik:

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) =$$

$$= a\left(x^2+2\cdot x\cdot\frac{b}{2a}+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right) =$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2+\underbrace{\frac{c}{a}-\frac{b^2}{4a^2}}_{\in\mathbb{R}}\right] =$$

$$= a\left[\left(x+\frac{b}{2a}\right)^2\pm d^2\right].$$

(#) Odrediti integrale

a) $\displaystyle\int\frac{dx}{x^2+4x+8}$; b) $\displaystyle\int\frac{7-8x}{2x^3-3x+1}\,dx$;

c) $\displaystyle\int\frac{3x-2}{x^2+6x+9}\,dx$; d) $\displaystyle\int\frac{6x^3-7x^2+3x-1}{2x-3x^2}\,dx$.

Rj a) $\displaystyle\int\frac{dx}{x^2+4x+8}$ Prvo primijetimo da je

$x^2+4x+8 = x^2+2\cdot x\cdot 2+4+4 = (x+2)^2+4$

i da je $d(x+2)=dx$. Prema tome

$$\int\frac{dx}{x^2+4x+8} = \int\frac{d(x+2)}{(x+2)^2+4} = \frac{1}{2}\operatorname{arctg}\frac{x+2}{2}+C$$

Prema formuli 8. $\displaystyle\int\frac{du}{u^2+a^2}=\frac{1}{a}\operatorname{arctg}\frac{u}{a}+C$ pri čemu $u=x+2$, $a=2$

b)

$$2x^3-3x+1 = 2\left(x^3-\frac{3}{2}x+\frac{1}{2}\right) = 2\left(x^2-2\cdot x\cdot\frac{3}{4}+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2+\frac{1}{2}\right)$$

$$= 2\left[\left(x-\frac{3}{4}\right)^2+\frac{1}{2}-\frac{9}{16}\right] = 2\left[\left(x-\frac{3}{4}\right)^2-\frac{1}{16}\right]$$

$$\int\frac{7-8x}{2x^3-3x+1}\,dx = \int\frac{7-8x}{2\left[\left(x-\frac{3}{4}\right)^2-\frac{1}{16}\right]}\,dx = \left|\begin{array}{ll} \text{uvodimo smjenu} & \\ x-\frac{3}{4}=t & x=t+\frac{3}{4} \\ dx=dt & 7-8x=7-8t-6 \\ & \quad =1-8t \end{array}\right|$$

$$= \frac{1}{2}\int\frac{1-8t}{t^2-\frac{1}{16}}\,dt = \frac{1}{2}\int\frac{dt}{t^2-\frac{1}{16}} - 4\int\frac{t\,dt}{t^2-\frac{1}{16}} =$$

$$= \frac{1}{2}\cdot\frac{1}{2\cdot\frac{1}{4}}\ln\left|\frac{t-\frac{1}{4}}{t+\frac{1}{4}}\right| - 4\int\frac{\frac{1}{2}\,d\left(t^2-\frac{1}{16}\right)}{t^2-1/16} =$$

$$= \ln\left|\frac{t-\frac{1}{4}}{t+\frac{1}{4}}\right| - 2\ln\left|t^2-\frac{1}{16}\right| + C =$$

$$= \ln\left|\frac{x-\frac{3}{4}-\frac{1}{4}}{t-\frac{3}{4}+\frac{1}{4}}\right| - 2\ln\left|\left(x-\frac{3}{4}\right)^2-\frac{1}{16}\right| + C$$

$$= \ln\left|\frac{x-1}{x-\frac{1}{2}}\right| - 2\ln\left|x^2-\frac{3}{2}x+\frac{1}{2}\right| + C$$

c) $\int \frac{3x-2}{x^2+6x+9}\,dx$ $\qquad x^2+6x+9 = x^2+2\cdot x\cdot 3+3^2 = (x+3)^2$

$$\int \frac{3x-2}{x^2+6x+9}\,dx = \int \frac{3x-2}{(x+3)^2}\,dx = \left| \begin{array}{l} \text{uvodimo smjenu} \\ x+3=t \\ dx=dt \\ x=t-3 \\ 3x-2=3t-9-2=3t-11 \end{array} \right| =$$

$$= \int \frac{3t-11}{t^2}\,dt = \int\left(\frac{3}{t}-\frac{11}{t^2}\right)dt = 3\int\frac{dt}{t} - 11\int t^{-2}dt =$$

$$= 3\ln|t| + 11t^{-1} + C = 3\ln|x+3| + \frac{11}{x+3} + C.$$

d) $J = \int \frac{6x^3-7x^2+3x-1}{2x-3x^2}\,dx$

Prvo podjelimo $6x^3-7x^2+3x-1$ sa $2x-3x^2$:

$$\begin{array}{l} (6x^3-7x^2+3x-1):(-3x^2+2x) = -2x+1 \\ -\ \ \underset{-}{6x^3}\underset{+}{-}4x^2 \\ \hline = \quad -3x^2+3x-1 \\ -\ \ \underset{+}{-}3x^2\ \underset{-}{+}2x \\ \hline = \qquad x-1 \end{array}$$

Prema tome

$$\frac{6x^3-7x^2+3x-1}{2x-3x^2} = -2x+1+\frac{x-1}{2x-3x^2}$$

$$\int \frac{6x^3-7x^2+3x-1}{2x-3x^2}\,dx = \int\left(-2x+1+\frac{x-1}{2x-3x^2}\right)dx =$$

$$= -2\int x\,dx + \int dx + \int\frac{(x-1)\,dx}{2x-3x^2} = -x^2+x+J_1$$

$$J_1 = -\frac{1}{3}\int\frac{(x-1)\,dx}{x^2-\frac{2}{3}x} = -\frac{1}{3}\int\frac{x}{x\left(x-\frac{2}{3}\right)}\,dx + \frac{1}{3}\int\frac{dx}{x^2-\frac{2}{3}x}$$

$$= \left| \begin{array}{l} x^2-\frac{2}{3}x = x^2-2\cdot x\cdot\frac{2}{6}+\left(\frac{2}{6}\right)^2-\left(\frac{2}{6}\right)^2 \\ \quad = \left(x-\frac{1}{3}\right)^2-\frac{1}{9} \end{array} \right| =$$

$$= -\frac{1}{3}\int\frac{d\left(x-\frac{2}{3}\right)}{x-\frac{2}{3}} + \frac{1}{3}\int\frac{d\left(x-\frac{1}{3}\right)}{\left(x-\frac{1}{3}\right)^2-\frac{1}{9}} =$$

$$= -\frac{1}{3}\ln\left|x-\frac{2}{3}\right| + \frac{1}{3}\cdot\frac{1}{2\cdot\frac{1}{3}}\ln\left|\frac{x-\frac{1}{3}-\frac{1}{3}}{x-\frac{1}{3}+\frac{1}{3}}\right| =$$

$$= -\frac{1}{3}\ln\left|x-\frac{2}{3}\right| + \underbrace{\frac{1}{2}\ln\left|\frac{x-\frac{2}{3}}{x}\right|}_{=\frac{1}{2}\ln\left|x-\frac{2}{3}\right|-\frac{1}{2}\ln|x|}$$

$$J = C - x^2 + x + \frac{1}{6}\ln\left|x-\frac{2}{3}\right| - \frac{1}{2}\ln|x|$$

traženo rješenje

(#) Odrediti integrale

a) $\int \frac{dx}{\sqrt{x^3-4x-3}}$ b) $\int \frac{(3x-5)\,dx}{\sqrt{9+6x-3x^2}}$

Rj:

a) $x^2-4x-3 = x^2-2\cdot x\cdot 2+2^2-2^2-3 = (x-2)^2-7$

$$\int \frac{dx}{\sqrt{x^3-4x-3}} = \int \frac{dx}{\sqrt{(x-2)^2-7}} = \int \frac{d(x-2)}{\sqrt{(x-2)^2-7}} =$$

$$= \ln\left|x-2+\sqrt{(x-2)^2-7}\right| + C$$

Prema formuli 11. $\int \frac{du}{\sqrt{u^2+a}} = \ln\left|u+\sqrt{u^2+a}\right| + C$ pri čemu je
$u = x-2,$
$a = -7$

b) $9+6x-3x^2 = (-3)\left[x^2-2x-3\right] = (-3)\left(x^2-2\cdot x\cdot 1+1^2-1^2-3\right) =$

$= (-3)\left[(x-1)^2-4\right] = 3\left[4-(x-1)^2\right]$

$$I = \int \frac{(3x-5)\,dx}{\sqrt{9+6x-3x^2}} = \int \frac{(3x-5)\,dx}{\sqrt{3\left[4-(x-1)^2\right]}} = \left| \begin{matrix} \text{uvodimo smjenu} & \\ z = x-1 & \\ dz = dx & 3x-5 = \\ x = z+1 & = 3z+3-5 \\ & = 3z-2 \end{matrix} \right| =$$

$$= \frac{1}{\sqrt{3}} \int \frac{3z-2}{\sqrt{4-z^2}}\,dz = \underbrace{\frac{3}{\sqrt{3}}}_{=\sqrt{3}} \underbrace{\int \frac{z\,dz}{\sqrt{4-z^2}}}_{=I_1} - \frac{2}{\sqrt{3}} \underbrace{\int \frac{dz}{\sqrt{4-z^2}}}_{=I_2}$$

$$I_1 = \int \frac{z\,dz}{\sqrt{4-z^2}} = \left| \begin{matrix} d(4-z^2) = -2z\,dz \\ z\,dz = -\frac{1}{2}\,d(4-z^2) \end{matrix} \right| = -\frac{1}{2}\int (4-z^2)^{-\frac{1}{2}}\,d(4-z^2) =$$

$$= -(4-z^2)^{\frac{1}{2}} + C_1$$

$$I_2 = \int \frac{dz}{\sqrt{4-z^2}} = \arcsin\frac{z}{2} + C_2$$

$$I = \sqrt{3}\left(-(4-z^2)^{\frac{1}{2}}\right) - \frac{2}{\sqrt{3}}\arcsin\frac{z}{2} + C =$$

$$= C - \sqrt{3(4-z^2)} - \frac{2}{\sqrt{3}}\arcsin\frac{z}{2} = \left| z = x-1 \right|$$

$$= C - \sqrt{9+6x-3x^2} - \frac{2}{\sqrt{3}}\arcsin\frac{x-1}{2}$$

(#) Pomoću formule za parcijalnu integraciju $\int u\,dv = uv - \int v\,du$ odrediti integrale

a) $\int \sqrt{t^2+b}\,dt$ i b) $\int \sqrt{a^2-t^2}\,dt$

pa uz pomoć dobijenog rezultata izračunati integrale

(i) $\int \sqrt{x^2-3}\,dx$

(ii) $\int \sqrt{x^2+2x+6}\,dx$

(iii) $\int \sqrt{3+4x-x^2}\,dx$

Rj.

a) $$I = \int \sqrt{t^2+b}\,dt = \left| \begin{matrix} u=\sqrt{t^2+b} & dv=dt \\ du=\frac{t\,dt}{\sqrt{t^2+b}} & v=t \end{matrix} \right| =$$

$$= t\sqrt{t^2+b} - \int \frac{t^2+b-b}{\sqrt{t^2+b}}\,dt = t\sqrt{t^2+b} - \int \frac{t^2+b}{\sqrt{t^2+b}}\,dt + b\int \frac{dt}{\sqrt{t^2+b}}$$

$$= t\sqrt{t^2+b} - \underbrace{\int \sqrt{t^2+b}\,dt}_{=I} + b\int \frac{dt}{\sqrt{t^2+b}} \Rightarrow$$

$$\Rightarrow 2I = t\sqrt{t^2+b} + b\int \frac{dt}{\sqrt{t^2+b}}$$

$$I = \frac{1}{2}t\sqrt{t^2+b} + \frac{1}{2}b\ln\left|t+\sqrt{t^2+b}\right| + C$$ traženo rješenje ...(*)

b) $$J = \int \sqrt{a^2-t^2}\,dt = \left| \begin{matrix} u=\sqrt{a^2-t^2} & dv=dt \\ du=\frac{-t}{\sqrt{a^2-t^2}}dt & v=t \end{matrix} \right| =$$

$$= t\sqrt{a^2-t^2} - \int \frac{-t^2 +a^2-a^2}{\sqrt{a^2-t^2}}\,dt =$$

$$= t\sqrt{a^2-t^2} - \int \frac{a^2-t^2}{\sqrt{a^2-t^2}} + a^2\int \frac{dt}{\sqrt{a^2-t^2}}$$

$$= t\sqrt{a^2-t^2} - \underbrace{\int \sqrt{a^2-t^2}}_{=J} + a^2\int \frac{dt}{\sqrt{a^2-t^2}}$$

$$2J = t\sqrt{a^2-t^2} + a^2\int \frac{dt}{\sqrt{a^2-t^2}}$$

$$J = \int \sqrt{a^2-t^2}\,dt = \frac{1}{2}t\sqrt{a^2-t^2} + \frac{1}{2}a^2\arcsin\frac{t}{a} + C$$ ...(**)

Izračunajmo sad date integrale

(i) $$\int \sqrt{x^2-3}\,dx = \left| \begin{matrix} \text{po formuli } (*) \\ \text{gdje je} \\ t=x,\ b=-3 \end{matrix} \right| = \frac{x}{2}\sqrt{x^2-3} - \frac{3}{2}\ln\left|x+\sqrt{a^2-3}\right| + C$$

(ii) $$x^2+2x+6 = x^2+2\cdot x\cdot 1+1^2-1^2+6 = (x+1)^2+5$$

$$\int \sqrt{x^2+2x+6}\,dx = \int \sqrt{(x+1)^2+5}\,d(x+1) = \left| \begin{matrix} \text{po formuli } (*) \\ \text{gdje je} \\ t=x+1,\ b=5 \end{matrix} \right| =$$

$$= \frac{x+1}{2}\sqrt{(x+1)^2+5} + \frac{5}{2}\ln\left|x+1+\sqrt{(x+1)^2+5}\right| + C$$

(iii) $$3+4x-x^2 = (-1)(x^2-4x-3) = (-1)(x^2-2\cdot x\cdot 2+2^2-2^2-3) = -\left[(x-2)^2-7\right] = 7-(x-2)^2$$

$$\int \sqrt{3+4x-x^2}\,dx = \int \sqrt{7-(x-2)^2}\,d(x-2) = \left| \begin{matrix} \text{po formuli } (**) \\ \text{gdje je } t=x-2,\ a^2=7 \end{matrix} \right| =$$

$$= \frac{x-2}{2}\sqrt{7-(x-2)^2} + \frac{7}{2}\arcsin\frac{x-2}{\sqrt{7}} + C.$$

Zadaci za vježbu

Odrediti integrale

① $\int \frac{dx}{x^2-x-6}$ ② $\int \frac{dx}{x^2+4x+29}$ ③ $\int \frac{dx}{4x-1-4x^2}$

④ $\int \frac{(4x-3)dx}{x^2+3x+4}$ ⑤ $\int \frac{(3x+4)dx}{x^2+5x}$ ⑥ $\int \frac{18x^2+13x}{1+6x+9x^2}dx$

⑦ $\int \frac{x^3-2x^2+4}{x^2+2x-3}dx$ ⑧ $\int \frac{dx}{\sqrt{2+x-x^2}}$ ⑨ $\int \frac{dx}{\sqrt{x^2-2x}}$

⑩ $\int \frac{(x+3)dx}{\sqrt{1-4x^2}}$ ⑪ $\int \frac{(x-3)dx}{\sqrt{x^2+6x}}$ ⑫* $\int \frac{x\,dx}{\sqrt{1-2x-3x^2}}$

⑬ $\int \sqrt{x^2+4x}\,dx$ ⑭ $\int \sqrt{1-2x-x^2}\,dx$

Rješenja:

1. $\frac{1}{5}\operatorname{arctg}\frac{x+2}{5}$ 2. ? 3. $\frac{1}{4x-2}$ 4. $2\ln(x^2+3x+4)-\frac{18}{\sqrt{7}}\operatorname{arctg}\frac{2x+3}{\sqrt{7}}$

5. $\frac{4}{5}\ln|x|+\frac{11}{5}\ln|x+5|$ 6. $2x+\frac{1}{9}\left(\frac{7}{3x+1}+\ln|3x+1|\right)$ 7. $\frac{1}{2}x^2-4x+$

$+\frac{3}{4}\ln|x-1|+\frac{41}{4}\ln|x+3|$ 8. $\arcsin\frac{2x-1}{3}$ 9. $\ln|x-1+\sqrt{x^2-2x}|$

10. $\frac{3}{2}\arcsin 2x-\frac{1}{4}\sqrt{1-4x^2}$ 11. $\sqrt{x^2+6x}-6\ln|x+3+\sqrt{x^2+6x}|$

12. $-\frac{1}{3\sqrt{3}}\arcsin\frac{3x+1}{2}-\frac{1}{3}\sqrt{1-2x-3x^2}$ 13. $\frac{x+2}{2}\sqrt{x^2+4x}-$

$-2\ln|x+2+\sqrt{x^2+4x}|$ 14. $\frac{x+1}{2}\sqrt{1-2x-x^2}+\arcsin\frac{x+1}{\sqrt{2}}$.

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija kvadratnog trinoma)

$\int\frac{dx}{ax^2+b}$, $\int\frac{dx}{\sqrt{ax^2+b}}$, smjena $\sqrt{|a|}\cdot x=\sqrt{|b|}\cdot t$

①

$$\int\frac{dx}{4x^2+9}=\int\frac{dx}{(2x)^2+3^2}=\left|\begin{matrix}2x=3t\\2dx=3dt\\dx=\frac{3}{2}dt\\t=\frac{2x}{3}\end{matrix}\right|=\frac{3}{2}\int\frac{dt}{(3t)^2+3^2}=\frac{3}{2}\int\frac{dt}{9t^2+9}=$$

$$=\frac{3}{2}\cdot\frac{1}{9}\int\frac{dt}{t^2+1}=\frac{1}{6}\operatorname{arctg}t+c=\frac{1}{6}\operatorname{arctg}\frac{2x}{3}+c$$

②

$$\int\frac{dx}{\sqrt{2x^2+25}}=\int\frac{dx}{\sqrt{(\sqrt{2}x)^2+5^2}}=\left|\begin{matrix}\sqrt{2}x=5t\\\sqrt{2}\,dx=5dt\\dx=\frac{5}{\sqrt{2}}dt\\t=\frac{\sqrt{2}}{5}x\end{matrix}\right|=\frac{5}{\sqrt{2}}\int\frac{dt}{\sqrt{\underbrace{25t^2+25}_{25\cdot(t^2+1)}}}=$$

$$=\frac{5}{\sqrt{2}}\cdot\frac{1}{5}\int\frac{dt}{\sqrt{t^2+1}}=\frac{1}{\sqrt{2}}\cdot\ln|t+\sqrt{t^2+1}|+c=\frac{\sqrt{2}}{2}\ln\left|\frac{\sqrt{2}}{5}x+\sqrt{\frac{2}{25}x^2+1}\right|+($$

③

$$\int\frac{dx}{5x^2-49}=\int\frac{dx}{(\sqrt{5}x)^2-7^2}=\left|\begin{matrix}\sqrt{5}x=7t\\\sqrt{5}dx=7dt\\dx=\frac{7}{\sqrt{5}}dt\\t=\frac{\sqrt{5}x}{7}\end{matrix}\right|=\frac{7}{\sqrt{5}}\int\frac{dt}{49t^2-49}=\frac{7}{\sqrt{5}}\cdot\frac{1}{49}\int\frac{dt}{t^2-1}$$

$$=\frac{1}{7\sqrt{5}}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+c=\frac{1}{14\sqrt{5}}\ln\left|\frac{\frac{\sqrt{5}}{7}x-1}{\frac{\sqrt{5}}{7}x+1}\right|+c=\frac{1}{14\sqrt{5}}\ln\left|\frac{\sqrt{5}x-7}{\sqrt{5}x+7}\right|$$

④

$$\int\frac{dx}{\sqrt{7-9x^2}}=\int\frac{dx}{\sqrt{(\sqrt{7})^2-(3x)^2}}=\left|\begin{matrix}3x=\sqrt{7}t\\3dx=\sqrt{7}dt\\dx=\frac{\sqrt{7}}{3}dt\\t=\frac{3x}{\sqrt{7}}\end{matrix}\right|=\frac{\sqrt{7}}{3}\int\frac{dt}{\sqrt{7-7t^2}}=\frac{\sqrt{7}}{3}\cdot\frac{1}{\sqrt{7}}\int\frac{dt}{\sqrt{1-t^2}}$$

$$=\frac{1}{3}\arcsin t+c=\frac{1}{3}\arcsin\left(\frac{3x}{\sqrt{7}}\right)+C$$

⑤ˇ $\int\frac{dx}{4x^2+11}$, Rj. $\frac{\sqrt{11}}{22}\operatorname{arctg}\frac{2\sqrt{11}x}{11}+C$

⑥ˇ $\int\frac{dx}{\sqrt{9x^2-16}}$, Rj. $\frac{1}{3}\ln|3x+\sqrt{9x^2-16}|+c$

$$\int \frac{dx}{ax^2+bx+c}, \quad \int \frac{dx}{\sqrt{ax^2+bx+c}}, \quad ax^2+bx+c = a(x-\alpha)^2+\beta$$

7. $I = \int \frac{dx}{x^2+6x+13}, \quad x^2+6x+13 = x^2+2\cdot x\cdot 3+3^2+4 = (x+3)^2+4$

$$I = \int \frac{dx}{(x+3)^2+2^2} = \left| \begin{array}{l} x+3=2t \\ dx=2dt \\ t=\frac{x+3}{2} \end{array} \right| = 2\int \frac{dt}{4t^2+4} = 2\cdot\frac{1}{4}\int\frac{dt}{t^2+1} = \frac{1}{2}\,\mathrm{arc\,tg}\, t + C$$

$$= \frac{1}{2}\,\mathrm{arctg}\,\frac{x+3}{2} + C$$

8. $I = \int \frac{dx}{\sqrt{2-x-x^2}}, \quad 2-x-x^2 = -x^2-x+2 = (-1)\left[x^2+x-2\right] =$

$$= (-1)\left(x^2+2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2-2\right) =$$

$$= (-1)\left[\left(x+\frac{1}{2}\right)^2-\frac{9}{4}\right] = \frac{9}{4}-\left(x+\frac{1}{2}\right)^2$$

$$I = \int \frac{dx}{\sqrt{\frac{9}{4}-(x+\frac{1}{2})^2}} = \int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x+\frac{1}{2}\right)^2}} = \left| \begin{array}{l} x+\frac{1}{2}=\frac{3}{2}t \\ dx=\frac{3}{2}dt \\ t=\frac{2x+1}{3} \end{array} \right| = \frac{3}{2}\int\frac{dt}{\sqrt{\frac{9}{4}-\frac{9}{4}t^2}} =$$

$$= \frac{3}{2}\cdot\frac{1}{\frac{3}{2}}\int\frac{dt}{\sqrt{1-t^2}} = \frac{3}{2}\cdot\frac{2}{3}\cdot \arcsin t + C = \arcsin\frac{2x+1}{3} + C$$

9. $I = \int \frac{dx}{2x^2-7x+3}, \quad 2x^2-7x+3 = 2\cdot\left(x^2-\frac{7}{2}x+\frac{3}{2}\right) = 2\cdot\left(x^2-2\cdot x\cdot\frac{7}{4}+\left(\frac{7}{4}\right)^2-\right.$

$$\left.-\left(\frac{7}{4}\right)^2+\frac{3}{2}\right) = 2\cdot\left[\left(x-\frac{7}{2}\right)^2+\frac{-49+24}{16}\right] = 2\left[\left(x-\frac{7}{2}\right)^2-\frac{25}{16}\right]$$

$$I = \frac{1}{2}\int\frac{dx}{(x-\frac{7}{2})^2-\frac{25}{16}} = \frac{1}{2}\int\frac{dx}{\left(x-\frac{7}{2}\right)^2-\left(\frac{5}{4}\right)^2} = \left| \begin{array}{l} x-\frac{7}{4}=\frac{5}{4}t \\ dx=\frac{5}{4}dt \\ t=\frac{4x-14}{5} \end{array} \right| = \frac{1}{2}\cdot\frac{5}{4}\int\frac{dt}{\frac{25}{16}t^2-\frac{25}{16}}$$

$$= \frac{\overset{1}{\cancel{5}}}{\cancel{8}_1}\cdot\frac{\overset{2}{\cancel{16}}}{\underset{5}{\cancel{25}}}\int\frac{dt}{t^2-1} = \frac{2}{5}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{5}\ln\left|\frac{\frac{4x-12}{5}-1}{\frac{4x-2}{5}+1}\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{\frac{4x-12}{5}}{\frac{4x-2}{5}}\right| + C = \frac{1}{5}\ln\left|\frac{4x-12}{4x-2}\right| + C = \frac{1}{5}\ln\left|\frac{2x-6}{2x-1}\right| + C$$

10. $^{\vee}$ $\int\frac{dx}{\sqrt{x^2+8x+25}}, \qquad$ Rj: $\ln\left|\frac{x+4}{3}+\sqrt{\left(\frac{x+4}{3}\right)^2+1}\right| + C$

$$\int\frac{mx+n}{ax^2+bx+c}\,dx, \quad \int\frac{mx+n}{\sqrt{ax^2+bx+c}}\,dx$$

11. $I = \int\frac{x+4}{x^2-4x+5}\,dx$

$(x^2-4x+5)' = 2x-4$

$x+4 = a\cdot(2x-4)+b, \quad a, b = ?$

$x+4 = 2ax-4a+b$

$2a=1 \qquad -4a+b=4$

$a=\frac{1}{2} \qquad -2+b=4$

$b=6$

$$I = \int\frac{\frac{1}{2}(2x-4)+6}{x^2-4x+5}\,dx$$

$$= \underbrace{\frac{1}{2}\int\frac{2x-4}{x^2-4x+5}\,dx}_{I_1} + \underbrace{6\int\frac{dx}{x^2-4x+5}}_{I_2} = \frac{1}{2}I_1+6I_2$$

$x^2-2\cdot x\cdot 2+2^2-2^2+5 =$
$= (x-2)^2+1$

$$I_1 = \left| \begin{array}{l} x^2-4x+5=t \\ (2x-4)dx=dt \end{array} \right| = \int\frac{dt}{t} = \ln|t|+C_1 = \ln|x^2-4x+5|+C_1$$

$$I_2 = \int\frac{dx}{(x-2)^2+1} = \left| \begin{array}{l} x-2=t \\ dx=dt \end{array} \right| = \int\frac{dt}{t^2+1} = \mathrm{arc\,tg}\, t + C_2 =$$

$$= \mathrm{arctg}(x-2)+C_2$$

$$I = \frac{1}{2}\ln|x^2-4x+5| + \mathrm{arctg}(x-2) + C$$

12. $I = \int\frac{x}{\sqrt{x^2+2x-5}}\,dx$

$(x^2+2x-5)' = 2x+2 \qquad 2a=1 \Rightarrow a=\frac{1}{2}$

$x = a(2x+2)+b \qquad 2a+b=0$

$x = 2ax+2a+b \qquad 2\cdot\frac{1}{2}+b=0$

$b=-1$

$$I = \int\frac{\frac{1}{2}(2x+2)-1}{\sqrt{x^2+2x-5}}\,dx = \frac{1}{2}\int\frac{2x+2}{\sqrt{x^2+2x-5}}\,dx - \int\frac{dx}{\sqrt{x^2+2x-5}} = \frac{1}{2}I_1 - I_2$$

$$\int\frac{2x+2}{\sqrt{x^2+2x-5}}\,dx = \left| \begin{array}{l} x^2-2x-5=t \\ (2x-2)dx=dt \end{array} \right| = \int\frac{dx}{\sqrt{t}} = \frac{1}{2}\sqrt{t}+C_1 = \frac{1}{2}\sqrt{x^2+2x-5}+C_1$$

$x^2+2x-5 = x^2+2\cdot x\cdot 1+1^2-1^2-5 =$
$= (x+1)^2-6$

$$I_2 = \int\frac{dx}{\sqrt{(x+1)^2-6}} = \left| \begin{array}{l} x+1=\sqrt{6}\,t \\ dx=\sqrt{6}\,dt \\ t=\frac{x+1}{\sqrt{6}} \end{array} \right| =$$

$$=\sqrt{6}\int\frac{dt}{\sqrt{6t^2-6}}=\frac{\sqrt{6}}{\sqrt{6}}\int\frac{dt}{\sqrt{t^2-1}}=\ln\left|t+\sqrt{t^2-1}\right|+C_2$$

$$=\ln\left|\frac{x+1}{\sqrt{6}}+\sqrt{\frac{(x+1)^2}{6}-1}\right|+C_2$$

$$I=\sqrt{x^2+2x-5}-\ln\left|\frac{x+1}{\sqrt{6}}+\sqrt{\frac{(x+1)^2}{6}-1}\right|+C$$

13. $I=\int\frac{6x-7}{x^2-4x+5}\,dx$, $(x^2-4x+5)'=2x-4$

$6x-7=a(2x-4)+b$

$6x-7=2ax-4a+b$

$2a=6 \Rightarrow a=3$

$b-4a=-7 \Rightarrow b=5$

$x^2-4x+5=x^2-2\cdot x\cdot 2+2^2+5-2^2=(x-2)^2+1$

$$I=\int\frac{3(2x-4)+5}{x^2-4x+5}\,dx=3\int\frac{2x-4}{x^2-4x+5}\,dx+5\int\frac{dx}{x^2-4x+5}=3I_1+5I_2$$

$$\int\frac{2x-4}{x^2-4x+5}\,dx=\left|\begin{matrix}x^2-4x+5=t\\(2x-4)\,dx=dt\end{matrix}\right|=\int\frac{dt}{t}=\ln|t|+C_1=\ln|x^2-4x+5|+C_1$$

$$I_2=\int\frac{dx}{(x-2)^2+1}=\left|\begin{matrix}x-2=t\\dx=dt\end{matrix}\right|=\int\frac{dt}{t^2+1}=\operatorname{arc\,tg} t+C_2=$$

$$=\operatorname{arc\,tg}(x-2)+C_2$$

$$I=3\ln|x^2-4x+5|+5\operatorname{arc\,tg}(x-2)+C$$

14. $\int\frac{3x+2}{\sqrt{x^2-8x-9}}\,dx$, Rj: $3\sqrt{x^2-8x-9}+14\ln\left|\frac{x-4}{5}+\sqrt{\left(\frac{x-4}{5}\right)^2-1}\right|+C$

15. $\int\frac{3x+4}{\sqrt{-x^2+6x-8}}\,dx$, Rj: $-3\sqrt{-x^2+6x-8}+13\arcsin(x-3)$

16. $\int\frac{2x+3}{\sqrt{4x^2+4x+3}}\,dx$

17. $\int\frac{x-4}{x^2-5x+6}\,dx$

Dio tablice integrala

1. $\int u^a\,du=\frac{u^{a+1}}{a+1}+C,\ a\neq-1.$

2. $\int u^{-1}\,du=\int\frac{du}{u}=\int\frac{u'}{u}\,dx=$ $=\ln|u|+C.$

3. $\int a^u\,du=\frac{a^u}{\ln a}+C;\ \int e^u\,du=$ $=e^u+C.$

4. $\int\sin u\,du=-\cos u+C.$

5. $\int\cos u\,du=\sin u+C.$

6. $\int\sec^2 u\,du=\operatorname{tg} u+C.$

7. $\int\operatorname{cosec}^2 u\,du=-\operatorname{ctg} u+C.$

8. $\int\frac{du}{u^2+a^2}=\frac{1}{a}\operatorname{arc\,tg}\frac{u}{a}+C.$

9. $\int\frac{du}{u^2-a^2}=\frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C.$

10. $\int\frac{du}{\sqrt{a^2-u^2}}=\arcsin\frac{u}{a}+C.$

11. $\int\frac{du}{\sqrt{u^2+a}}=$ $=\ln\left|u+\sqrt{u^2+a}\right|+C.$

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6. Integracija trigonometriskih f-ja

Dosta često se pojavljuju integrali izraza, koji sadrže trigonometriske f-je sljedećih tipova:

I. $\int \sin^n x\,dx$, $\int \cos^n x\,dx$ II. $\int \sin^m x \cos^n x\,dx$

III. $\int tg^n x\,dx$, $\int ctg^n x\,dx$

gdje su m i n - pozitivni cijeli brojevi,

IV. $\int \sin ax \cos bx\,dx$, $\int \sin ax \sin bx\,dx$,

$\int \cos ax \cos bx\,dx$

koji se mogu svesti na formulu integriranja, a prematome i naći, tako što će se slijediti neko od pravila:

1. Integrali od parnog stepena sinusa ili kosinusa mogu se odrediti pomoću smanjivanja stepena (dvaput) pomoću formula:

$$\sin^2 u = \frac{1}{2}(1-\cos 2u); \quad \cos^2 u = \frac{1}{2}(1+\cos 2u); \quad \sin u \cos u = \frac{1}{2}\sin 2u.$$

2. Integrale neparnog stepena od sinusa ili kosinusa možemo odrediti putem razdvajanja jednog od drugog faktora i zamjeniti komplementarnu f-ju novom promjenjivom.

3. Integrale tipa II možemo odrediti po pravilu 1, ako su oba broja m i n parna, ili po pravilu 2 ako su m ili n (ili oba) neparna.

4. Integrale tipa III možemo odrediti putem zamjene tgx, ili dosljedno, ctgx novom promjenjivom.

5. Integrale tipa IV možemo odrediti tako što ćemo razložiti podintegralni izraz na dijelove pomoću formula

$$\sin ax \cos bx = \frac{1}{2}[\sin(a+b)x + \sin(a-b)x]$$

$$\sin ax \sin bx = \frac{1}{2}[\cos(a-b)x - \cos(a+b)x]$$

$$\cos ax \cos bx = \frac{1}{2}[\cos(a+b)x + \cos(a-b)x]$$

(#) Odrediti integrale

a) $\int \sin^2 3x\,dx$; b) $\int \cos^4 x\,dx$; c) $\int \sin^5 x\,dx$;

d) $\int \sin^4 x \cos^2 x\,dx$; e) $\int \sin^6 kx \cos^3 kx\,dx$; f) $\int \sin^3 x \cos^5 x\,dx$.

Rj.

a) Prema pravilu 1 imamo

$$\int \sin^2 3x\,dx = \frac{1}{2}\int (1-\cos 6x)\,dx = \frac{1}{2}\int dx - \frac{1}{2}\cdot\frac{1}{6}\int \cos 6x\,d(6x) =$$

$$= \frac{x}{2} - \frac{1}{12}\sin 6x + C.$$

b) Prema pravilu 1 imamo

$$\int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx = \frac{1}{4}\int (1+\cos 2x)^2\,dx =$$

$$= \frac{1}{4}\left[\int dx + 2\int \cos 2x \cdot \frac{1}{2}\,d(2x) + \underbrace{\int \cos^2 2x\,dx}_{I_1}\right]$$

$$I_1 = \int \cos^2 2x\,dx = \frac{1}{2}\int (1+\cos 4x)\,dx = \frac{1}{2}\int dx + \frac{1}{2}\cdot\frac{1}{4}\int \cos 4x\,d(4x) =$$

$$= \frac{1}{2}x + \frac{1}{8}\sin 4x$$

Prema tome

$$\int \cos^4 x\,dx = \frac{1}{4}\left(x + \sin 2x + \frac{x}{2} + \frac{1}{8}\sin 4x\right) + C$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

c) $$\int \sin^5 x\,dx = \left|\begin{array}{l}\text{prema pravilu 2}\\ \sin^5 x = \sin^4 x \cdot \sin x\end{array}\right| = \int (\sin^2 x)^2 \sin x\,dx =$$

$$= \int (1-\cos^2 x)^2 \sin x\,dx = \left|\begin{array}{l}\cos x = z\\ -\sin x\,dx = dz\\ \sin x\,dx = -dz\end{array}\right| = \int (1-z^2)^2(-dz) =$$

$$= -\int (1-2z^2+z^4)\,dz = -z + \frac{2}{3}z^3 - \frac{1}{5}z^5 + C =$$

$$= C - \cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x$$

d) $$\int \sin^4 x \cos^2 x\,dx = \left|\begin{array}{l}\text{prema pravilu 3}\\ \sin u \cos u = \frac{1}{2}\sin 2u\\ \sin^2 = \frac{1}{2}(1-\cos 2\ )\end{array}\right| =$$

$$= \int \sin^2 x \sin^2 x \cos^2 x\,dx = \int \sin^2 x\,(\sin x \cos x)^2\,dx =$$

$$= \int \frac{1}{2}(1-\cos 2x)\cdot\frac{1}{4}\sin^2 2x\,dx = \frac{1}{8}\underbrace{\int \sin^2 2x\,dx}_{I_1} -$$

$$- \frac{1}{8}\underbrace{\int \sin^2 2x \cos 2x\,dx}_{I_2} = \frac{1}{8}\left(I_1 - I_2\right)$$

$$I_1 = \int \sin^2 2x\,dx = \frac{1}{2}\int (1-\cos 4x)\,dx = \frac{1}{2}\int dx - \frac{1}{8}\int \cos 4x\,d(4x)$$

$$= \frac{1}{2}x - \frac{1}{8}\sin 4x$$

$$I_2 = \int \sin^2 2x \cos 2x\,dx = \left|\begin{array}{l}\sin 2x = z\\ 2\cos 2x\,dx = dz\end{array}\right| = \frac{1}{2}\int z^2\,dz = \frac{1}{2}\cdot\frac{z^3}{3}$$

$$= \frac{1}{6}\sin^3 2x.$$

Prema tome

$$\int \sin^4 x \cos^2 x \, dx = \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x - \frac{1}{6}\sin^3 2x\right) + C$$

e) $$\int \sin^6 kx \cos^3 kx \, dx = \left| \begin{array}{l} \text{prema pravilu 3 pišemo} \\ \cos^3 kx = \cos^2 kx \cdot \cos kx \\ \quad = (1-\sin^2 kx)\cos kx \end{array}\right| =$$

$$= \int \sin^6 kx\,(1-\sin^2 kx)\cos kx\,dx = \left|\begin{array}{l}\sin kx = z\\ k\cos kx\,dx = dz\end{array}\right| =$$

$$= \frac{1}{k}\int z^6(1-z^2)\,dz = \frac{1}{k}\left(\int z^6\, dz - \int z^8\, dz\right) =$$

$$= \frac{1}{k}\left(\frac{z^7}{7} - \frac{z^9}{9}\right) + C = \frac{1}{7k}\sin^7 kx - \frac{1}{9k}\sin^9 kx + C$$

f) $$\int \sin^3 x \cos^5 x\,dx = \left|\begin{array}{l}\text{prema pravilu 3}\\ \sin^3 x = \sin^2 x \sin x\\ \quad = (1-\cos^2 x)\sin x\end{array}\right| =$$

$$= \int (1-\cos^2 x)\cos^5 x \sin x\,dx = \left|\begin{array}{l}\cos x = z\\ -\sin x\,dx = dz\end{array}\right| =$$

$$= -\int(1-z^2)z^5\, dz = -\int z^5\, dz + \int z^7\, dz = \frac{1}{8}z^8 - \frac{1}{6}z^6 + C$$

$$= \frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + C.$$

(#) Odrediti integrale

a) $\int tg^4 x\,dx$ b) $\int \sin 3x \cos 5x\,dx$

Rj.

a) $$\int tg^4 x\,dx = \left|\begin{array}{l}\text{primjenom pravila 4}\\ tg\,x = z\\ x = arc\,tg\,z\\ dx = \dfrac{dz}{1+z^2}\end{array}\right| = \int\frac{z^4}{z^2+1}\,dz \overset{(*)}{=}$$

$$\begin{array}{l} z^4 : (z^2+1) = z^2 - 1 \\ -\underline{z^4 \underset{-}{+} z^2} \\ \quad -z^2 \\ -\underline{\;-z^2 \underset{+}{-} 1} \\ \qquad 1 \end{array} \qquad \frac{z^4}{z^2+1} = z^2 - 1 + \frac{1}{z^2+1}$$

$$\overset{(*)}{=} \int\left(z^2 - 1 + \frac{1}{z^2+1}\right)dz = \frac{1}{3}z^3 - z + arc\,tg\,z + C =$$

$$= \frac{1}{3}tg^3 x - tg\,x + x + C.$$

b) $$\int \sin 3x\cos 5x\,dx = \left|\begin{array}{l}\text{primjenom pravila 5}\\ \sin 3x\cos 5x = \frac{1}{2}[\sin 8x + \sin(-2x)]\end{array}\right| =$$

$$= \frac{1}{2}\int[\sin 8x - \sin 2x]\,dx = \frac{1}{2}\cdot\frac{1}{8}\int\sin 8x\,d(8x) -$$

$$- \frac{1}{2}\cdot\frac{1}{2}\int\sin 2x\,d(2x) = -\frac{1}{16}\cos 8x + \frac{1}{4}\cos 2x + C$$

Zadaci za vježbu

Odrediti integrale

① $\int \cos^2 5x \, dx$ ② $\int \cos^5 x \, dx$ ③ $\int \sin^2 x \cos^2 x \, dx$

④ $\int \sin^3 x \cos^2 x \, dx$ ⑤ $\int \sin^3 x \cos^3 x \, dx$ ⑥ $\int \sin^4 x \, dx$

⑦ $\int \mathrm{ctg}^4 y \, dy$ ⑧ $\int \cos \frac{4}{3}x \cos 3x \, dx$ ⑨ $\int \sin 5x \sin 6x \, dx$

⑩ $\int \sin at \cos bt \, dt$ ⑪* $\int \sin 3x \sin 4x \sin 5x \, dx$

⑫* $\int (\mathrm{tg}\, z + \mathrm{ctg}\, z)^3 \, dz$

Rješenja:

1° $\frac{x}{2} + \frac{1}{20} \sin 10x$ 2° $\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5} \sin^5 x$ 3° $\frac{x}{8} - \frac{1}{32} \sin 4x$

4° $\frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x$ 5° $\frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x$ 6° $\frac{3}{8} x - \frac{1}{4} \sin 2x +$

$+ \frac{1}{32} \sin 4x$ 7° $y + \mathrm{ctg}\, y - \frac{1}{3} \mathrm{ctg}^3 y$ 8° $\frac{3}{26} \sin \frac{13}{3} x + \frac{3}{10} \sin \frac{5}{3} x$

9° $\frac{1}{2} \sin x - \frac{1}{22} \sin 11x$ 10° $\frac{\cos(a-b)t}{2(b-a)} - \frac{\cos(a+b)t}{2(a+b)}$

11° $\frac{\cos 12x}{48} - \frac{\cos 6x}{24} - \frac{\cos 4x}{16} - \frac{\cos 2x}{8}$

12° $\frac{1}{2}(\mathrm{tg}^2 z - \mathrm{ctg}^2 z) + 2 \ln|\mathrm{tg}\, z|$.

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija trigonometriskih funkcija)

$$\int \sin^m x \cdot \cos^n x \, dx \quad (m, n \in \mathbb{N}_0)$$

ako je m neparan uvodimo smjenu $\cos x = t$

ako je n neparan uvodimo smjenu $\sin x = t$

ako su m i n parni koristimo formule

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

① $\int \sin^3 x \cdot \cos^{12} x \, dx = \int \sin x \cdot \sin^2 x \cdot \cos^{12} x \, dx = \int \underline{\sin x} (1 - \cos^2 x) \cos^{12} x \, \underline{dx}$

$= \left| \begin{array}{l} \cos x = t \\ -\sin x \, dx = dt \\ \sin x \, dx = -dt \end{array} \right| = \int (1 - t^2) \cdot t^{12} \cdot (-dt) = -\int (t^{12} - t^{14}) \, dt = -\int t^{12} dt +$

$+ \int t^{14} dt = -\frac{t^{13}}{13} + \frac{t^{15}}{15} + C = -\frac{1}{13} \cos^{13} x + \frac{1}{15} \cos^{15} x + C$

② $\int \sin^4 x \cdot \cos^5 x \, dx = \int \sin^4 x \cdot \cos^4 x \cdot \cos x \, dx = \int \sin^4 x (\cos^2 x)^2 \cos x \, dx$

$= \int \sin^4 x (1 - \sin^2 x)^2 \cos x \, dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \end{array} \right| = \int t^4 \cdot (1 - t^2)^2 \, dt =$

$= \int t^4 (1 - 2t^2 + t^4) \, dt = \int (t^8 - 2t^6 + t^4) \, dt = \frac{t^9}{9} - 2 \cdot \frac{t^7}{7} + \frac{t^5}{5} + C$

$= \frac{1}{9} \sin^9 x - \frac{2}{7} \sin^7 x + \frac{1}{5} \sin^5 x + C$

③$^\vee$ $\int \sin^7 x \cdot \cos^{10} x \, dx$ ⑤$^\vee$ $\int \sin^5 x \, dx$

Rj. $-\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$

④$^\vee$ $\int \sin^2 x \cdot \cos^3 x \, dx$

⑥ $\int \cos^4 x \, dx = \int (\cos^2 x)^2 dx = \int \left(\frac{1 + \cos 2x}{2} \right)^2 dx =$

$$=\int\frac{1+2\cos2x+\cos^2 2x}{4}dx=\frac14\int dx+\frac24\int\cos2x\,dx+\frac14\int\cos^2 2x\,dx=$$

$$=\frac14 x+\frac12\cdot\frac12\sin2x+\frac14\int\frac{1+\cos4x}{2}dx=\frac14x+\frac14\sin2x+\frac18\left(\int dx+\int\cos4x\,dx\right)$$

$$\left[\int\cos2x\,dx-\left|\begin{matrix}2x=t\\2dx=dt\\dx=\frac{dt}{2}\end{matrix}\right|=\int\cos t\cdot\frac{dt}{2}=\frac12\sin t+C=\frac12\sin2x+C\right]$$

$$=\frac14x+\frac14\sin2x+\frac18x+\frac18\cdot\frac14\sin4x+C=\frac38x+\frac14\sin2x+\frac1{32}\sin4x+C$$

Napomena: Zadatak možemo riješiti i parcijalnom integracijom $\int\cos^4x\,dx=\int\cos^3x\cdot\cos x\,dx=\left|\begin{matrix}u=\cos^3x & dv=\cos x\,dx\\ \vdots & \end{matrix}\right|$

7. $$\int\sin^2x\cdot\cos^2x\,dx=\int\left(\frac{1-\cos2x}{2}\right)\left(\frac{1+\cos2x}{2}\right)dx=\frac14\int(1-\cos^2 2x)dx$$

$$=\frac14\int dx-\frac14\int\cos^2 2x\,dx=\frac14x-\frac14\int\frac{1+\cos4x}{2}dx=\frac14x-\frac18\int(1+\cos4x)\,dx$$

$$=\frac14x-\frac18\int dx-\frac18\int\cos4x\,dx=\frac14x-\frac18x-\frac18\cdot\frac14\sin4x+C=$$

$$=\frac18x-\frac1{32}\sin4x+C$$

8.$^{\vee}$ $\int\sin^6x\,dx$

11.$^{\vee}$ $\int\sin^4 2x\,dx$

Rj. $\frac38x-\frac18\sin4x+\frac1{64}\sin8x+C$

9.$^{\vee}$ $\int\sin^2x\cdot\cos^4x\,dx$

10.$^{\vee}$ $\int\cos^6x\,dx$

Rj. $\frac5{16}x-\frac1{48}\sin^3 2x+\frac14\sin2x+\frac3{64}\sin4x+C$

$$\int\sin\alpha x\cdot\sin\beta x\,dx,\quad\int\sin\alpha x\cdot\cos\beta x\,dx,\quad\int\cos\alpha x\cdot\cos\beta x\,dx$$

koristimo formule:

$$\sin\alpha x\cdot\sin\beta x=\frac12\left[\cos(\alpha-\beta)x-\cos(\alpha+\beta)x\right]$$

$$\sin\alpha x\cdot\cos\beta x=\frac12\left[\sin(\alpha+\beta)x+\sin(\alpha-\beta)x\right]$$

$$\cos\alpha x\cdot\cos\beta x=\frac12\left[\cos(\alpha+\beta)x+\cos(\alpha-\beta)x\right]$$

12. $$\int\sin4x\cdot\sin2x\,dx=\frac12\int(\cos2x-\cos6x)\,dx=\frac12\int\cos2x\,dx-$$

$$-\frac12\int\cos6x\,dx=\frac12\cdot\frac12\sin2x-\frac12\cdot\frac16\sin6x+C=\frac14\sin2x-\frac1{12}\sin6x+C$$

13. $$\int\sin3x\cdot\cos5x\,dx=\frac12\int(\sin8x+\sin(-2x))\,dx=\frac12\int\sin8x\,dx-$$

$$-\frac12\int\sin2x\,dx=\frac12\cdot\left(-\frac18\right)\cos8x-\frac12\cdot\left(-\frac12\right)\cos2x+C=-\frac1{16}\cos8x+\frac14\cos2x+C$$

14. $$\int\cos x\cdot\cos3x\cdot\cos5x\,dx=\int\frac12(\cos4x+\cos(-2x))\cos5x\,dx=$$

$$=\frac12\int(\cos4x+\cos2x)\cos5x\,dx=\frac12\int\cos4x\cos5x\,dx+\frac12\int\cos2x\cos5x\,dx$$

$$=\frac12\cdot\frac12\int(\cos9x+\cos x)\,dx+\frac12\cdot\frac12\int(\cos7x+\cos3x)\,dx=$$

$$=\frac14\int\cos9x\,dx+\frac14\int\cos x\,dx+\frac14\int\cos7x\,dx+\frac14\int\cos3x\,dx=$$

$$=\frac14\cdot\frac19\sin9x+\frac14\sin x+\frac14\cdot\frac17\sin7x+\frac14\cdot\frac13\sin3x+C$$

$$=\frac1{36}\sin9x+\frac14\sin x+\frac1{28}\sin7x+\frac1{12}\sin3x+C$$

15.$^{\vee}$ $\int\sin x\cdot\sin2x\cdot\sin4x\,dx$ Rj. $-\frac1{20}\cos5x-\frac1{12}\cos3x+\frac1{28}\cos7x+$

$+\frac14\cos x+C$

(16.)$^{\vee}$ $\int \sin 2x \cdot \cos 3x \cdot \sin 5x \, dx$

(17.) $\int \sin^2 \frac{x}{2} \cos 3x \, dx = \int \frac{1-\cos 2\cdot\frac{x}{2}}{2} \cos 3x \, dx = \frac{1}{2}\int (1-\cos x)\cdot \cos 3x \, dx$

$= \frac{1}{2}\int (\cos 3x - \cos x \cdot \cos 3x) \, dx = \frac{1}{2}\int \cos 3x \, dx - \frac{1}{2}\int \cos x \cos 3x \, dx$

$= \frac{1}{2}\cdot\frac{1}{3}\sin 3x - \frac{1}{2}\cdot\frac{1}{2}\int (\cos 4x + \cos 2x) \, dx = \frac{1}{6}\sin 3x - \frac{1}{4}\int \cos 4x \, dx$

$- \frac{1}{4}\int \cos 2x \, dx = \frac{1}{6}\sin 3x - \frac{1}{16}\sin 4x - \frac{1}{8}\sin 2x + C$

(18.)$^{\vee}$ $\int \sin^2 2x \cdot \cos^2 3x \, dx$

(19.)$^{\vee}$ $\int \sin^3 x \cdot \cos^2 2x \, dx$

(#) Izračunati integral $I = \int \frac{8\cos x - \sin x}{2\cos x + \sin x} \, dx$,

R: $(2\cos x + \sin x)' = -2\sin x + \cos x$

$$\frac{8\cos x - \sin x}{2\cos x + \sin x} = A + B\frac{-2\sin x + \cos x}{2\cos x + \sin x} \quad /\cdot(2\cos x + \sin x)$$

$$8\cos x - \sin x = A(2\cos x + \sin x) + B(-2\sin x + \cos x)$$

$$8\cos x - \sin x = (2A + B)\cos x + (A - 2B)\sin x$$

$$\begin{array}{l} 2A + B = 8 \\ A - 2B = -1 \quad /\cdot 2 \\ \hline 2A + B = 8 \\ -\ 2A - 4B = -2 \\ \hline 5B = 10 \\ B = 2 \end{array} \qquad \begin{array}{l} 2A + B = 8 \\ 2A = 8 - 2 \\ 2A = 6 \\ A = 3 \end{array}$$

$$I = \int \frac{8\cos x - \sin x}{2\cos x + \sin x} \, dx = \int \left(3 + 2\frac{-2\sin x + \cos x}{2\cos x + \sin x}\right) dx =$$

$$= 3\int dx + 2\int \frac{-2\sin x + \cos x}{2\cos x + \sin x} \, dx = \left| \begin{array}{l} 2\cos x + \sin x = t \\ (-2\sin x + \cos x)dx = dt \end{array} \right|$$

$$= 3x + 2\int \frac{dt}{t} = 3x + 2\ln|t| + C =$$

$$= 3x + 2\ln|2\cos x + \sin x| + C$$

7. Integracija racionalnih f-ja

Funkciju oblika $\frac{P(x)}{Q(x)}$, gdje su $P(x)$ i $Q(x)$ polinomi, nazivamo RACIONALNA F-JA. Ako je stepen polinoma $P(x)$ manji od stepena polinoma $Q(x)$, tada kažemo da je f-ja $\frac{P(x)}{Q(x)}$ PRAVA RACIONALNA F-JA. Ako je stepen polinoma $P(x)$ veći ili jednak od stepena polinoma $Q(x)$, tada je funkcija $\frac{P(x)}{Q(x)}$ NEPRAVA RACIONALNA F-JA.

Ako je racionalna f-ja $\frac{P(x)}{Q(x)}$ neprava, tada dijeljenjem polinoma $P(x)$ sa polinomom $Q(x)$ dobijemo količnik polinom $K(x)$, i ostatak dijeljenja, polinom $R(x)$, tako da je

$$P(x)=Q(x)\cdot K(x)+R(x) \quad \text{tj.} \quad \frac{P(x)}{Q(x)}=K(x)+\frac{R(x)}{Q(x)}$$

Kako je stepen polinoma ostatka $R(x)$ manji od stepena polinoma djelioca $Q(x)$, slijedi da je $\frac{R(x)}{Q(x)}$ prava racionalna f-ja.

Pravu racionalnu f-ju možemo integrirati metodom neodređenih koeficijenata. To činimo na sljedeći način: Najprije polinom u nazivniku racionalne f-je rastavimo na proste faktore oblika

$$(x-a)^n \quad \text{i} \quad (x^2+px+q)^n \quad \text{gdje su}$$

$$n\in N,\ a,p,q\in R,\ p^2-4q<0.$$

Svakom faktoru oblika $(x-a)^n$ pridružimo f-ju oblika

$$\frac{A_1}{x-a}+\frac{A_2}{(x-a)^2}+\dots+\frac{A_n}{(x-a)^n}$$

($A_1, A_2, \dots, A_n$ su konstante koje trebamo odrediti),

a svakom faktoru oblika $(x^2+px+q)^n$ pridružimo f-ju oblika

$$\frac{M_1x+N_1}{x^2+px+q}+\frac{M_2x+N_2}{(x^2+px+q)^2}+\dots+\frac{M_nx+N_n}{(x^2+px+q)^n}$$

gdje su $M_1, M_2, \dots, M_n$ i $N_1, N_2, \dots, N_n$ konstante koje treba odrediti.

Navedene racionalne f-je nazivamo parcijalni razlomci. Prema tome, svaku pravu racionalnu f-ju najprije rastavimo na zbir parcijalnih razlomaka, a zatim svaki od njih posebno integriramo. Kod neprave racionalne f-je najprije vršimo dijeljenje polinoma u brojniku sa polinomom u nazivniku,

(#) Odrediti integrale

a) $\int \frac{3x^2+8}{x^3+4x^2+4x}\,dx$; b) $\int \frac{2x^5+6x^3+1}{x^4+3x^2}\,dx$;

c) $\int \frac{x^3+4x^2+4x}{x^4+x}\,dx$; d) $\int \frac{(x^3-3)\,dx}{x^4+10x^2+25}$.

Rj.

a) $\int \frac{3x^2+8}{x^3+4x^2+4x}\,dx$

Nazivnik rastavljamo na faktore.

$$x^3+4x^2+4x = x(x^2+4x+4) = x(x+2)^2$$

Poslije ovoga podintegralnu f-ju rastavljamo na sumu

$$\frac{3x^2+8}{x(x+2)^2} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$$

Da bi odredili koeficijente A, B, C čitavu jednakost množimo sa $x(x+2)^2$ nakon čega dobijamo

$$3x^2+8 = A\underbrace{(x+2)^2}_{x^2+2x+2} + Bx(x+2) + Cx$$

$$= (A+B)x^2 + (4A+2B+C)x + 4A$$

Sad izjednačavamo koeficijente koji stoje uz x^2, x i x^0.

x^2: $A+B=3$ $\Rightarrow B=1$

x: $4A+2B+C=0$ $\Rightarrow C=-10$

x^0: $\underline{4A=8}$ $\Rightarrow A=2$

Prema tome dobili smo $\frac{3x^2+8}{x(x+2)^2} = \frac{2}{x} + \frac{1}{x+2} - \frac{10}{(x+2)^2}$

Sad možemo riješiti dati integral

$$\int \frac{3x^2+8}{x^3+4x^2+4x}\,dx = \int\left(\frac{2}{x} + \frac{1}{x+2} - \frac{10}{(x+2)^2}\right)dx =$$

$$= 2\int\frac{dx}{x} + \int\frac{dx}{x+2} - 10\int (x+2)^{-2}\,d(x+2) =$$

$$= 2\ln|x| + \ln|x+2| + \frac{10}{x+2} + C$$

b) $\int \frac{2x^5+6x^3+1}{x^4+3x^2}\,dx$

Podjelimo $2x^5+6x^3+1$ sa x^4+3x^2.

$$(2x^5+6x^3+1):(x^4+3x^2)=2x$$

$$\underline{-\;2x^5+6x^3\qquad}$$

$$1$$

Prema tome

$$\frac{2x^5+6x^3+1}{x^4+3x^2} = 2x + \frac{1}{x^4+3x^2}$$

Rastavimo nazivnik na faktore

$$x^4+3x^2 = x^2(x^2+3)$$

Napišimo ostatak $\frac{1}{x^4+3x^2}$ u obliku sume

$$\frac{1}{x^2(x^2+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+3}$$

Da bi smo odredili koeficijente A, B, C i D pomnožimo dobijenu jednakost sa $x^2(x^2+3)$:

$$1 = Ax(x^2+3) + B(x^2+3) + (Cx+D)x^2 =$$

$$= (A+C)x^3 + (B+D)x^2 + 3Ax + 3B$$

Izjednačimo brojeve koji stoje uz x^3, x^2, x i x^0:

x^3: $A+C=0 \quad \Rightarrow C=0$

x^2: $B+D=0 \quad \Rightarrow D=-\frac{1}{3}$

x: $3A=0 \quad \Rightarrow A=0$

x^0: $3B=1 \quad \Rightarrow B=\frac{1}{3}$

$$\frac{1}{x^2(x^2+3)} = \frac{1}{3x^2} - \frac{1}{3(x^2+3)}$$

Sad nije teško izračunati dati integral

$$\int \frac{2x^5+6x^3+1}{x^4+3x^2}\,dx = \int \left(2x + \frac{1}{x^4+3x^2}\right) dx =$$

$$= \int \left(2x + \frac{\frac{1}{3}}{x^2} - \frac{\frac{1}{3}}{x^2+3}\right) dx = 2\int x\,dx + \frac{1}{3}\int x^{-2}\,dx$$

$$- \frac{1}{3}\int \frac{dx}{x^2+3} = x^2 - \frac{1}{3x} - \frac{1}{3\sqrt{3}}\,\mathrm{arc\,tg}\,\frac{x}{\sqrt{3}} + C$$

c) $$\int \frac{x^3+4x^2+4x}{x^4+x}\,dx$$

Rastavimo nazivnik x^4+x na faktore

$$x^4+x = x(x^3+1) = x(x+1)(x^2-x+1)$$

$$\frac{x^3+4x^2-2x+1}{x(x+1)(x^2-x+1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1} \quad /\cdot x(x+1)(x^2-x+1)$$

$$x^3+4x^2-2x+1 = A(x^3+1) + Bx(x^2-x+1) + (Cx+D)(x^2+x)$$

$$= (A+B+C)x^3 + (C+D-B)x^2 + (B+D)x + A$$

$A+B+C=1$

$-B+C+D=4$

$B+D=-2$

$A=1$

$A=1;\ B=-2;\ C=2;\ D=0$

$$\frac{x^3+4x^2-2x+1}{x(x+1)(x^2-x+1)} = \frac{1}{x} - \frac{2}{x+1} + \frac{2x}{x^2-x+1}$$

Sad nije teško odrediti dati integral

$$I = \int \frac{x^3+4x^2-2x+1}{x^4+x}\,dx = \int \frac{dx}{x} - 2\int \frac{dx}{x+1} + 2\int \frac{x\,dx}{x^2-x+1}$$

$$= \ln|x| - 2\ln|x+1| + 2I_1$$

Da bi odredili I_1 imamo

$$x^2-x+1 = x^2 - 2\cdot x\cdot\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1 = \left(x-\frac{1}{2}\right)^2 + \frac{3}{4}$$

$$I_1 = \int \frac{x\,dx}{(x-\frac{1}{2})^2+\frac{3}{4}} = \left| \begin{array}{l} x-\frac{1}{2}=t \\ dx=dt \end{array} \right| = \int \frac{(t+\frac{1}{2})\,dt}{t^2+\frac{3}{4}} = \frac{1}{2}\int \frac{d(t^2+\frac{3}{4})}{t^2+\frac{3}{4}}$$

$$+ \frac{1}{2}\int \frac{dt}{t^2+\frac{3}{4}} = \frac{1}{2}\ln\left(t^2+\frac{3}{4}\right) + \frac{1}{\sqrt{3}}\,\mathrm{arc\,tg}\,\frac{2t}{\sqrt{3}} =$$

$$= \frac{1}{2}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\,\mathrm{arc\,tg}\,\frac{2x-1}{\sqrt{3}}\ .$$

Prema tome imamo

$$I = \ln\frac{|x|(x^2-x+1)}{(x+1)^2} + \frac{2}{\sqrt{3}} \qquad \frac{2x-1}{\sqrt{3}} + C\ .$$

d) $\int \frac{(x^3-3)\,dx}{x^4+10x^2+25}$

$$x^4+10x^2+25=(x^2+5)^2$$

$$\frac{x^3-3}{(x^2-5)^2}=\frac{Ax+B}{x^2+5}+\frac{Cx+D}{(x^2+5)^2} \quad /\cdot(x^2-5)^2$$

$$x^3-3=(Ax+B)(x^2+5)+(Cx+D)$$

$$=Ax^3+Bx^2+(5A+C)x+(5B+D)$$

$$\begin{array}{l} A=1 \\ B=0 \\ 5A+C=0 \\ 5B+D=-3 \end{array} \Rightarrow \begin{array}{l} A=1 \\ B=0 \\ C=-5 \\ D=-3 \end{array}$$

$$\frac{x^3-3}{(x^2+5)^2}=\frac{x}{x^2+5}-\frac{5x+3}{(x^2+5)^2}$$

Sad imamo

$$\int\frac{x^3-3}{x^4+10x^2+25}\,dx=\underbrace{\int\frac{x\,dx}{x^2+5}}_{I_1}-5\underbrace{\int\frac{x\,dx}{(x^2+5)^2}}_{I_2}-3\underbrace{\int\frac{dx}{(x^2+5)^2}}_{I_3}$$

$$I_1=\int\frac{x\,dx}{x^2+5}=\frac{1}{2}\int\frac{d(x^2+5)}{x^2+5}=\frac{1}{2}\ln|x^2+5|$$

$$I_2=\int\frac{x\,dx}{(x^2+5)^2}=\frac{1}{2}\int(x^2+5)^{-2}d(x^2+5)=\frac{1}{2}\cdot\frac{(x^2+5)^{-1}}{-1}=$$

$$=-\frac{1}{2(x^2+5)}$$

$$I_3=\int\frac{dx}{(x^2+5)^2}=\left|\begin{array}{l} x=\sqrt{5}\,tg\,z=\sqrt{5}\,\frac{\sin z}{\cos z} \\ dx=\sqrt{5}\,\frac{\cos^2 z+\sin^2 z}{\cos^2 z}dz=\frac{\sqrt{5}}{\cos^2 z}dz \\ x^2=5\,\frac{1-\cos^2 z}{\cos^2 z} \end{array}\right|=$$

$$=\int\frac{\frac{\sqrt{5}\,dz}{\cos^2 z}}{\left(\frac{5-5\cos^2 z}{\cos^2 z}+5\right)^2}=\sqrt{5}\int\frac{\frac{dz}{\cos^2 z}}{\frac{25}{\cos^4 z}}=\int\cos^2 z\,dz=$$

$$=\frac{\sqrt{5}}{25}\cdot\frac{1}{2}\int(1+\cos 2z)\,dz=\frac{\sqrt{5}}{50}\left(z+\frac{1}{2}\sin 2z\right)$$

$$=\frac{1}{10\sqrt{5}}\left(arc\,tg\,\frac{x}{\sqrt{5}}+\frac{x\sqrt{5}}{x^2+5}\right)$$

Prema tome

$$I=\frac{1}{2}\ln(x^2+5)+\frac{5}{2(x^2+5)}-\frac{3}{10\sqrt{5}}\left(arc\,tg\,\frac{x}{\sqrt{5}}+\frac{x\sqrt{5}}{x^2+5}\right)+C$$

$$=\frac{1}{2}\ln(x^2+5)+\frac{25-3x}{10(x^2+5)}-\frac{3}{10\sqrt{5}}\,arc\,tg\,\frac{x}{\sqrt{5}}+C$$

traženo
rješenje

Zadaci za vježbu

Odrediti integrale

① $\displaystyle\int \frac{dx}{x^3-x^2}$ ② $\displaystyle\int \frac{dx}{x^3+x}$ ③ $\displaystyle\int \frac{x\,dx}{x^3-1}$

④ $\displaystyle\int \frac{(x^2+1)\,dx}{x^3-3x^2+3x-1}$ ⑤ $\displaystyle\int \frac{(7x-15)\,dx}{x^3-2x^2+5x}$

⑥ $\displaystyle\int \frac{2t^5-2t+1}{1-t^4}\,dt$ ⑦ $\displaystyle\int \frac{z^2\,dz}{z^4+5z^2+4}$

⑧ $\displaystyle\int \frac{x^4\,dx}{x^4-2x^2+1}$ ⑨* $\displaystyle\int \frac{(x+1)\,dx}{x^4+4x^2+4}$

⑩* $\displaystyle\int \frac{1-x^4}{1+x^4}\,dx$

Rjesenja:

1. $\frac{1}{x}+\ln\left|1-\frac{1}{x}\right|$ 2. $\ln\frac{|x|}{\sqrt{x^2+1}}$ 3. $\frac{1}{3}\ln\frac{|x-1|}{\sqrt{x^2+x+1}}+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x+1}{\sqrt{3}}$

4. $\ln|x-1|-\frac{2x-1}{(x-1)^2}$ 5. $3\ln\frac{\sqrt{x^2-2x+5}}{|x|}+2\operatorname{arctg}\frac{x-1}{2}$

6. $\frac{1}{4}\ln\left|\frac{1+t}{1-t}\right|-t^2+\frac{1}{2}\operatorname{arctg}t$ 7. $\frac{2}{3}\operatorname{arctg}\frac{z}{2}-\frac{1}{3}\operatorname{arctg}z$

8. $\frac{2x^3-3x}{2(x^2-1)}+\frac{3}{4}\ln\left|\frac{x-1}{x+1}\right|$ 9. $\frac{x-2}{4(x^2+2)}+\frac{1}{4\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}$

10. $-x+\frac{1}{2\sqrt{2}}\ln\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x\sqrt{2}}{1-x^2}$

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija racionalnih funkcija)

$q_m(x)$ - polinom stepena m

npr. $p(x)=5x^7-3x^2+x+8$, polinom 7-og stepena

Racionalna f-ja je količnik dva polinoma.

$s(x)=\dfrac{p_n(x)}{q_m(x)}$. Za $n<m$ $s(x)$ je prava racionalna f-ja

Racionalnu f-ju razlažemo na proste razlomke. Prosti razlomci su oblika:

$$\frac{A}{(ax+b)^n},\quad \frac{Bx+C}{(ax^2+bx+c)^n},\quad n\in\mathbb{N}$$

Izračunajte integrale:

① $$I=\int\frac{x}{(x-1)(x+1)^2}\,dx$$

Rj. $$\frac{x}{(x-1)(x+1)^2}=\frac{a}{x-1}+\frac{b}{x+1}+\frac{c}{(x+1)^2}\quad /\cdot(x-1)(x+1)^2$$

$$x=a(x+1)^2+b(x-1)(x+1)+c(x-1)$$
$$x=a(x^2+2x+1)+b(x^2-1)+c(x-1)$$
$$x=(a+b)x^2+(2a+c)x+(a-b-c)$$

$$\Rightarrow \begin{array}{ll} a+b=0 & (1)\\ 2a+c=1 & (2)\\ a-b-c=0 & (3)\\ \hline (1)\quad a+b=0 & \\ (2)+(3):\ 3a-b=1 & \\ \hline 4a=1 & \end{array}\qquad a=\frac{1}{4}\quad b=-\frac{1}{4}\quad c=\frac{1}{2}$$

$$\frac{x}{(x-1)(x+1)^2}=\frac{\frac{1}{4}}{x-1}+\frac{-\frac{1}{4}}{x+1}+\frac{\frac{1}{2}}{(x+1)^2}$$

$$I=\frac{1}{4}\int\frac{dx}{x-1}-\frac{1}{4}\int\frac{dx}{x+1}+\frac{1}{2}\int\frac{dx}{(x+1)^2}=$$

$$=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|+\frac{1}{2}\,\frac{(x+1)^{-1}}{-1}+C=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2(x+1)}+C$$

2. $I=\int\frac{8x-31}{x^2-9x+14}\,dx$,

$x^2-9x+14=0$ $\quad x_{1,2}=\frac{9\pm5}{2}$

$D=81-56$ $\quad x_1=2$, $x_2=7$

$D=25$ $\quad x^2-9x+14=(x-2)(x-7)$

Rj. $\frac{8x-31}{x^2-9x+14}=\frac{8x-31}{(x-2)(x-7)}=\frac{a}{x-2}+\frac{b}{x-7}\quad /(x-2)(x-7)$

$8x-31=a(x-7)+b(x-2)$

$8x-31=(a+b)x+(-7a-2b)$

$$\begin{array}{l} a+b=8 \quad /\cdot 2\\ -7a-2b=-31\\ \hline 2a+2b=16\\ -7a-2b=-31\\ \hline \end{array}\qquad \begin{array}{l}-5a=-15\\ a=3\\ b=5\end{array}$$

$$\frac{8x-31}{x^2-9x+14}=\frac{3}{x-2}+\frac{5}{x-7}$$

$$I=3\int\frac{dx}{x-2}+5\int\frac{dx}{x-7}=3\ln|x-2|+5\ln|x-7|+C$$

3. $I=\int\frac{dx}{x^3-2x^2+x}$, $\quad x^3-2x^2+x=x(x^2-2x+1)=x(x-1)^2$

Rj. $\frac{1}{x^3-2x^2+x}=\frac{1}{x(x-1)^2}=\frac{a}{x}+\frac{b}{x-1}+\frac{c}{(x-1)^2}\quad /x(x-1)^2$

$1=a(x-1)^2+bx(x-1)+cx$

$1=a(x^2-2x+1)+b(x^2-x)+cx$

$1=(a+b)x^2+(-2a-b+c)x+a$

$$\begin{array}{l} a+b=0\\ -2a-b+c=0\\ a=1\\ \hline b=-1 \Rightarrow c=1\end{array}$$

$$\frac{1}{x^3-2x^2+x}=\frac{1}{x}+\frac{-1}{x-1}+\frac{1}{(x-1)^2}$$

$$I=\int\frac{dx}{x}-\int\frac{dx}{x-1}+\int\frac{dx}{(x-1)^2}=\ln|x|-\ln|x-1|-\frac{1}{x-1}+C$$

4. $I=\int\frac{x^3+x+1}{x^4-1}\,dx$, $\quad x^4-1=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1)$

Rj. $\frac{x^3+x+1}{x^4-1}=\frac{a}{x-1}+\frac{b}{x+1}+\frac{cx+d}{x^2+1}\quad /(x-1)(x+1)(x^2+1)$

$x^3+x+1=a(x+1)(x^2+1)+b(x-1)(x^2+1)+(cx+d)(x-1)(x+1)$

$x^3+x+1=a(x^3+x+x^2+1)+b(x^3+x-x^2-1)+(cx+d)(x^2-1)$

$x^3+x+1=a(x^3+x^2+x+1)+b(x^3-x^2+x-1)+c(x^3-x)+d(x^2-1)$

$x^3+x+1=(a+b+c)x^3+(a-b+d)x^2+(a+b-c)x+(a-b-d)$

$$\begin{array}{llr} a+b+c & =1 & (1)\\ a-b\quad +d & =0 & (2)\\ a+b-c & =1 & (3)\\ a-b\quad -d & =1 & (4)\\ \hline \end{array}$$

(1)-(4): $2b+c+d=0$

(2)-(4): $2d=-1 \Rightarrow d=-\frac{1}{2}$

(3)-(4) $2b-c+d=0$

$$\begin{array}{l} 2b+c=\frac12\\ -\;2b-c=\frac12\\ \hline 2c=0 \Rightarrow c=0\end{array}$$

$2b=\frac12$

$b=\frac14$

$a=1-\frac14-0$

$a=\frac34$

$$\frac{x^3+x+1}{x^4-1}=\frac{\frac34}{x-1}+\frac{\frac14}{x+1}+\frac{-\frac12}{x^2+1}$$

$$I=\frac34\int\frac{dx}{x-1}+\frac14\int\frac{dx}{x+1}-\frac12\int\frac{dx}{x^2+1}=\frac34\ln|x-1|+\frac14\ln|x+1|-\frac12\,\text{arc tg}\,x+C$$

5. $\int\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)}\,dx$,

$x^2+1=0$ $\quad x^2-x+1=0$

$D=-4<0$ $\quad D=1-4<0$

x^2+1 ; x^2-x+1 se ne mogu rastaviti

Rj. $\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)}=\frac{ax+b}{x^2+1}+\frac{cx+d}{x^2-x+1}\quad /(x^2+1)(x^2-x+1)$

$x^3+3x-5=(ax+b)(x^2-x+1)+(cx+d)(x^2+1)$

$x^3+3x-5=a(x^3-x^2+x)+b(x^2-x+1)+c(x^3+x)+d(x^2+1)$

$x^3+3x-5=(a+c)x^3+(-a+b+d)x^2+(a-b+c)x+(b+d)$

$$\begin{array}{llr} a+\quad c & =1 & (1)\\ -a+b\quad +d & =0 & (2)\\ a-b+c & =3 & (3)\\ b\quad +d & =-5 & (4)\\ \hline \end{array}$$

(1) $a+\quad c=1$

(2)-(4) $-a=5 \Rightarrow a=-5$

$$\begin{array}{l}(3)\ a-b+c=3\\ \hline c=6\end{array}$$

$-5-b+6=3 \Rightarrow b=-2$ $\quad\Rightarrow d=-3$

$$\frac{x^3-3x-5}{(x^2+1)(x^2-x+1)}=\frac{-5x-2}{x^2+1}+\frac{6x-3}{x^2-x+1}$$

$$I=\int\frac{-5x-2}{x^2+1}\,dx+3\int\frac{2x-1}{x^2-x+1}\,dx=I_1+3I_2$$

$(x^2+1)'=2x$

$-5x-2=\alpha\cdot2x+\beta$

$-5x-2=2\cdot\frac{-5}{2}x-2$

$$I_1 = \int \frac{2\cdot\frac{-5}{2}x-2}{x^2+1}\,dx = -\frac{5}{2}\int\frac{2x}{x^2+1}\,dx - 2\int\frac{dx}{x^2+1} = -\frac{5}{2}\ln|x^2+1| - 2\,\mathrm{arctg}(x^2+1)+C_1$$

$$I_2 = \int\frac{2x-1}{x^2-x+1}\,dx = \left|\begin{matrix} x^2-x+1=t \\ (2x-1)\,dx = dt\end{matrix}\right| = \int\frac{dt}{t} = \ln|x^2-x+1| + C_2$$

$$I = -\frac{5}{2}\ln|x^2+1| - 2\,\mathrm{arctg}(x^2+1) + 3\ln|x^2-x+1| + C$$

⑥ $I = \int \frac{3}{x(x+1)^3}\,dx$

Rj. $\frac{3}{x(x+1)^3} = \frac{a}{x} + \frac{b}{x+1} + \frac{c}{(x+1)^2} + \frac{d}{(x+1)^3} \quad /\cdot x(x+1)^3$

$$3 = a(x+1)^3 + bx(x+1)^2 + cx(x+1) + dx$$

Ako uvrstimo $x=0$ u gornju jednakost dobićemo $3=a$

tj. $a=3$: $\quad 3 = 3(x+1)^3 + bx(x+1)^2 + cx(x+1) + dx$

za $x=-1$ imamo $\quad 3 = d\cdot(-1) \Rightarrow d=-3$

$$3 = 3(x+1)^3 + bx(x+1)^2 + cx(x+1) - 3x$$

za $x=1$: $\quad 3 = 3\cdot 2^3 + b\cdot 1\cdot 2^2 + c\cdot 1\cdot 2 - 3\cdot 1 \Rightarrow 4b+2c=-18$

za $x=-2$: $\quad 3 = 3\cdot(-1)^3 + b(-2)(-1)^2 + c(-2)(-1) - 3\cdot(-2) \Rightarrow -2b+2c=0$ } +

$\Rightarrow b = c = -3$

$$\int\frac{dx}{(x+1)^3} = \int(x+1)^{-3}\,dx = \frac{(x+1)^{-2}}{-2} + C_1 = -\frac{1}{2(x+1)^2} + C_1$$

$$\frac{3}{x(x+1)^3} = \frac{3}{x} + \frac{-3}{x+1} + \frac{-3}{(x+1)^2} + \frac{-3}{(x+1)^3}$$

$$I = 3\int\frac{dx}{x} - 3\int\frac{dx}{x+1} - 3\int\frac{dx}{(x+1)^2} - 3\int\frac{dx}{(x+1)^3} = 3\ln|x| - 3\ln|x+1| + \frac{3}{x+1} + \frac{3}{2(x+1)^2} + C$$

⑦ ✓ $\int\frac{2x-3}{(x^2-3x+2)^3}\,dx \qquad$ Rj. $-\frac{1}{2(x^2-3x+2)^2} + c$

⑧ ✓ $\int\frac{x^3+x+1}{x(x^2+1)}\,dx \qquad$ Rj. $x + \ln\left|\frac{x}{\sqrt{x^2+1}}\right| + c$

⑨ ✓ $\int\frac{x^4}{x^4-1}\,dx \qquad$ Rj. $x + \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\,\mathrm{arctg}\,x + c$

⑩ ✓ $\int\frac{dx}{(x^2-4x+3)(x^2+4x+5)} \qquad$ Rj: $\frac{1}{52}\ln|x-3| - \frac{1}{20}\ln|x-1| + \frac{1}{65}\ln(x^2+4x+5) + \frac{7}{130}\,\mathrm{arctg}(x+2) + c$

⑪ $I = \int\frac{-x^5-2x^2+2}{(x-1)(x^2+2x+1)}\,dx$

Rj. $(x-1)(x^2+2x+1) = x^3+2x^2+x-x^2-2x-1 = x^3+x^2-x-1$

$$(-x^5-2x^2+2):(x^3+x^2-x-1) = -x^2+x-2-\frac{x}{x^3+x^2-x-1}$$

$$\underline{-\ -x^5-x^4+x^3+x^2}$$

$$\underline{\quad /\!/ \quad x^4-x^3-3x^2+2}$$

$$\underline{-\quad x^4+x^3-x^2-x}$$

$$/\!/ \quad -2x^3-2x^2+x+2$$

$$\underline{-\quad -2x^3-2x^2+2x+2}$$

$$-x$$

$$I = \int\left(-x^2+x-2-\frac{x}{(x-1)(x^2+2x+1)}\right)dx$$

$$= -\int x^2\,dx + \int x\,dx - 2\int dx - \int\frac{x}{(x-1)(x+1)^2}\,dx$$

$= -\frac{x^3}{3} + \frac{x^2}{2} - 2x - I_1$, integral I_1 smo odredili u zadatku 1

$$I = -\frac{x^3}{3} + \frac{x^2}{2} - 2x - \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{2(x+1)} + c$$

⑫ $I = \int\frac{x^5-2x^3+x-1}{x^3-2x^2+x}\,dx$

Rj. $(x^5-2x^3+x-1):(x^3-2x^2+x) = x^2+2x+1-\frac{1}{x^3-2x^2+x}$

$$\underline{-\quad x^5-2x^4+x^3}$$

$$/\!/ \quad 2x^4-3x^3+x-1$$

$$\underline{-\quad 2x^4-4x^3+2x^2}$$

$$/\!/ \quad x^3-2x^2+x-1$$

$$\underline{-\quad x^3-2x^2+x}$$

$$-1$$

$$I = \int\left(x^2+2x+1-\frac{1}{x^3-2x^2+x}\right)dx$$

$$= \int x^2\,dx + 2\int x\,dx + \int dx - \int\frac{dx}{x(x-1)^2}$$

$= \frac{x^3}{3} + 2\cdot\frac{x^2}{2} + x - I_2$, integral I_2 smo odredili u zadatku 3.

$$I = \frac{x^3}{3} + x^2 + x - \ln|x| + \ln|x-1| + \frac{1}{x-1} + C$$

⑬ ✓ $I = \int\frac{x^5+2x^3-4}{(x-2)(x^2+x+1)}\,dx \qquad$ Rj. $I = \frac{x^3}{8} + \frac{x^2}{2} + 4x + \frac{44}{7}\ln|x-2| + \frac{5}{14}\ln|x^2+x+1| + \frac{11}{7\sqrt{3}}\,\mathrm{arctg}\frac{2x+1}{\sqrt{3}} + c$

⑭ $$\int \frac{x^5-60x^3+73x^2+171}{x^2-9x+14}\,dx$$

Rj. $$(x^5-60x^3+73x^2+171):(x^2-9x+14)=x^3+9x^2+7x+10+\frac{-8x+31}{x^2-9x+14}$$

$$\begin{array}{l}
-\ \underline{x^5-9x^4+14x^3}\\
=\ 9x^4-74x^3+73x^2+171\\
-\ \underline{9x^4-81x^3+126x^2}\\
=\ 7x^3-53x^2+171\\
-\ \underline{7x^3-63x^2+98x}\\
=\ 10x^2-98x+171\\
-\ \underline{10x^2-90x+140}\\
\quad -8x+31
\end{array}$$

$$I=\int\left(x^3+9x^2+7x+10+\frac{-8x+31}{x^2-9x+14}\right)dx$$

$$=\int x^3dx+9\int x^2dx+7\int x\,dx+10\int dx-\int\frac{8x-31}{x^2-9x+14}\,dx$$

$I=\frac{x^4}{4}+9\cdot\frac{x^3}{3}+7\,\frac{x^2}{2}+10x-I_3$, integral I_3 smo odredili u zadatku broj 2.

$$I=\frac{1}{4}x^4+3x^3+\frac{7}{2}x^2+10x-3\ln|x-2|+5\ln|x-7|+c$$

⑮ $$I=\int \frac{x^7-2x^6+x^5+x^4+2x^2}{x^4-1}\,dx$$

Rj. $$(x^7-2x^6+x^5+x^4+2x^2):(x^4-1)=x^3-2x^2+x+1+\frac{x^3+x+1}{x^4-1}$$

$$\begin{array}{l}
-\ \underline{x^7-x^3}\\
=\ -2x^6+x^5+x^4+x^3+2x^2\\
-\ \underline{-2x^6+2x^2}\\
=\ x^5+x^4+x^3\\
\quad \underline{x^5-x}\\
\quad x^4+x^3+x\\
-\ \underline{x^4-1}\\
\quad x^3+x+1
\end{array}$$

$$I=\int\left(x^3-2x^2+x+1+\frac{x^3+x+1}{x^4-1}\right)dx=$$

$$=\int x^3dx-2\int x^2dx+\int x\,dx+\int dx+\int\frac{x^3+x+1}{x^4-1}\,dx$$

$I=\frac{x^4}{4}-2\cdot\frac{x^3}{3}+\frac{x^2}{2}+x+I_4$, integral I_4 smo odredili u zadatku 4

$$I=\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{1}{2}x^2+x+\frac{3}{4}\ln|x-1|+\frac{1}{4}\ln|x+1|-\frac{1}{2}\operatorname{arctg}x+c$$

⑯^{v} $$\int\frac{x^5+2x^3+4x+4}{x^4+2x^3+2x^2}\,dx$$ Rj. $$I=\frac{x^2}{2}-2x+\frac{2}{x}+2\ln|x^2+2x+2|-$$
$$-2\operatorname{arctg}(x+1)+c$$

---

⑰ $$\int\frac{-2x^7-x^6-x^3+6x^2-x}{(x^2+1)(x^2-x+1)}\,dx$$

Rj. $$(x^2+1)\cdot(x^2-x+1)=x^4-x^3+x^2+x^2-x+1=x^4-x^3+2x^2-x+1$$

$$(-2x^7-x^6-x^3+6x^2-x):(x^4-x^3+2x^2-x+1)=-2x^3-3x^2+x+5+\frac{x^3+3x-5}{x^4-x^3+2x^2-x+1}$$

$$\begin{array}{l}
-\ \underline{-2x^7+2x^6-4x^5+2x^4-2x^3}\\
=\ -3x^6+4x^5-2x^4+x^3+6x^2-x\\
-\ \underline{-3x^6+3x^5-6x^4+3x^3-3x^2}\\
=\ x^5+4x^4-2x^3+9x^2-x\\
-\ \underline{x^5-x^4+2x^3-x^2+x}\\
=\ 5x^4-4x^3+10x^2-2x\\
-\ \underline{5x^4-5x^3+10x^2-5x+5}\\
\quad x^3+3x-5
\end{array}$$

$$I=\int\left(-2x^3-3x^2+x+5+\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)}\right)dx$$

$$=-2\int x^3dx-3\int x^2dx+\int x\,dx+5\int dx+\int\frac{x^3+3x-5}{(x^2+1)(x^2-x+1)}\,dx$$

$I=-2\cdot\frac{x^4}{4}-3\cdot\frac{x^3}{3}+\frac{x^2}{2}+5x+I_5$, integral I_5 smo odredili u zadatku broj 5

$$I=-\frac{1}{2}x^4-x^3+\frac{1}{2}x^2+5x-\frac{5}{2}\ln|x^2+1|-2\operatorname{arctg}(x^2+1)+3\ln|x^2-x+1|+c$$

⑱^{v} $$\int\frac{x^5+x^4-8}{x^3-4x}$$

⑲^{v} $$\int\frac{x^5+2}{x^3-1}\,dx$$

8. Integracija nekih iracionalnih f-ja

Ovu lekciju možemo podjeliti na pet vrsta integrala

I. $\int R(x, x^{\alpha}, x^{\beta}, \dots)\,dx$ gdje je R racionalna f-ja,

$\alpha = \frac{m_1}{n_1}$, $\beta = \frac{m_2}{n_2}$ -uvodimo smjenu $x = t^k$ gdje je k broj takav da u novodobijenom integralu na promjenjivoj t ostanu samo cijeli stepeni.

$\int R(x, (ax+b)^{\alpha}, (ax+b)^{\beta}\,dx$ ili

$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\alpha}, \left(\frac{ax+b}{cx+d}\right)^{\beta}, \dots\right)$

rješavamo uvođenjem smjene $ax+b = t^k$ ili

$\frac{ax+b}{cx+d} = t^k$.

II. $\int R(x, \sqrt{a^2-x^2})\,dx$ -uvodimo smjenu $x = a\sin t$;

$\int R(x, \sqrt{a^2+x^2})\,dx$ -uvodimo smjenu $x = a\,tg\,t$;

$\int R(x, \sqrt{x^2-a^2})\,dx$ -uvodimo smjenu $x = \frac{a}{\cos t}$

III. $\int x^m (a+bx^n)^p\,dx$ (integral binomnog diferencijala)

a) kada je $p \in \mathbb{Z}$ (p cijeli broj) -uvodimo smjenu $x = t^s$ gdje je $s = NZS(m, n)$ (najmanji zajednički sadržalac)

ili (ako je $p \in \mathbb{Z}$) razlažemo na dijelove pomoću binomne formule

b) kada je $\frac{m+1}{n} \in \mathbb{Z}$ -uvodimo smjenu $a+bx^n = t^r$ gdje je r nazivnik od p

c) kada je $\frac{m+1}{n} + p \in \mathbb{Z}$ -uvodimo smjenu $a+bx^n = x^n t^r$ gdje je r nazivnik broja p

IV. $\int \frac{P_n(x)}{\sqrt{V}}\,dx$ gdje je $P_n(x)$ polinom n-tog stepena, a $V = ax^2+bx+c$. Ovaj integral možemo odrediti po formuli

$$\int \frac{P_n(x)}{\sqrt{V}}\,dx = (A_1 x^{n-1} + A_2 x^{n-2} + \dots + A_n)\sqrt{V} + B\int \frac{dx}{\sqrt{V}}$$

gdje su $A_1, A_2, \dots, A_n, B$ brojevi koje dobijemo iz sistema jednačina, a sistem jednačina dobijemo tako što datu formulu prvo diferenciramo a onda dobijeni diferencijal pomnožimo sa $\sqrt{V}$. (Metoda Ostrogradski)

V $\int \frac{(Ax+B)\,dx}{(x-\alpha)\sqrt{ax^2+bx+c}}$ -uvodimo smjenu $x-\alpha = \frac{1}{t}$

(#) Odrediti integrale

a) $\int \frac{1+\sqrt[4]{x}}{x+\sqrt{x}}\,dx$ b) $\int \frac{1}{x^2}\sqrt{\frac{1+x}{x}}\,dx$ c) $\int \frac{\sqrt{(4-x^2)^3}}{x^6}\,dx$

d) $\int \frac{dx}{x^2\sqrt[3]{(1+x^3)^5}}$ e) $\int \frac{2x^2-x-5}{\sqrt{x^2-2x}}\,dx$ f) $\int \frac{dx}{(x-1)\sqrt{1-x^2}}$

Rj.

a) $$\int \frac{1+\sqrt[4]{x}}{x+\sqrt{x}}\,dx = \left| \begin{array}{l} x=t^4 \\ dx=4t^3dt \end{array} \right| = \int \frac{1+t}{t^4+t^2}\,4t^3dt$$

$$= 4\int \frac{t^2+t^{\,+1-1}}{t^2+1}\,dt = 4\int\left(1+\frac{t-1}{t^2+1}\right)dt =$$

$$= 4\left(\int dt + \int \frac{\overbrace{\frac{1}{2}d(t^2+1)}^{=t\,dt}}{t^2+1} - \int\frac{dt}{t^2+1}\right) =$$

$$= 4t + 2\ln(t^2+1) - 4\,\mathrm{arc\,tg}\,t + C$$

$$= 4\sqrt[4]{x} + 2\ln(1+\sqrt{x}) - 4\,\mathrm{arctg}\sqrt[4]{x} + C$$

b) $$\int \frac{1}{x^2}\sqrt{\frac{1+x}{x}}\,dx$$ Prema pravilu I: $\frac{1+x}{x}=t^2$

Odavde imamo $\frac{1\cdot x-(1+x)\cdot 1}{x^2}\,dx = 2t\,dt \Rightarrow \frac{-1}{x^2}\,dx = 2t\,dt$

$\Rightarrow \frac{dx}{x^2} = -2t\,dt$

$$\boxed{\frac{1+x}{x}=t^2 \Rightarrow 1+x=t^2\cdot x \Rightarrow (1-t^2)x=-1 \Rightarrow x=\frac{1}{t^2-1}}$$

$$\int \frac{1}{x^2}\sqrt{\frac{1+x}{x}}\,dx = \left| \begin{array}{l} \frac{1+x}{x}=t^2 \\ \frac{dx}{x^2}=-2t\,dt \end{array} \right| = \int (-2)\,t\cdot t\,dt =$$

$$= -2\int t^2dt = -\frac{2}{3}t^3+C = -\frac{2}{3}\sqrt{\left(\frac{1+x}{x}\right)^3}+C$$

c) $$\int \frac{\sqrt{(4-x^2)^3}}{x^6}\,dx = \left| \begin{array}{l} x=2\sin t \\ dx=2\cos t\,dt \end{array} \right| = \int \frac{\sqrt{(4-4\sin^2 t)^3}}{64\sin^6 t}\,2\cos t\,dt$$

$\nearrow \sqrt{4^3(1-\sin^2 t)^3}$

$$= \frac{2^3\cdot 2}{64}\int \frac{\cos^3 t}{\sin^6 t}\cos t\,dt = \frac{1}{4}\int \frac{\cos^4 t}{\sin^6 t}\,dt = \frac{1}{4}\int \mathrm{ctg}^4 t\cdot\frac{dt}{\sin^2 t}$$

$$= \left| \begin{array}{l} d(\mathrm{ctg}\,t) = \frac{-dt}{\sin^2 t} \\ \frac{dt}{\sin^2 t} = -d(\mathrm{ctg}\,t) \end{array} \right| = -\frac{1}{4}\int \mathrm{ctg}^4 t\,d(\mathrm{ctg}\,t) =$$

$$= -\frac{1}{4}\cdot\frac{1}{5}\mathrm{ctg}^5 t + C = \left| \begin{array}{l} \mathrm{ctg}^5 t = \frac{\cos^5 t}{\sin^5 t} = \frac{(\cos^2 t)^{\frac{5}{2}}}{(\sin^2 t)^{\frac{5}{2}}} = \\ = \frac{(1-\sin^2 t)^{\frac{5}{2}}}{(\sin^2 t)^{\frac{5}{2}}} = \frac{\left(1-\frac{x^2}{4}\right)^{\frac{5}{2}}}{\left(\frac{x^2}{4}\right)^{\frac{5}{2}}} = \\ = \frac{(4-x^2)^{\frac{5}{2}}}{(x^2)^{\frac{5}{2}}} = \frac{\sqrt{(4-x^2)^5}}{x^5} \end{array} \right| = C - \frac{\sqrt{(4-x^2)^5}}{20x^5}$$

d) $$\int \frac{dx}{x^2\sqrt[3]{(1+x^3)^5}}$$ ovo je integral binomnog diferencijala $m=-2,\ n=3,\ p=-\frac{5}{3}$

$\frac{m+1}{n} = \frac{-2+1}{3} = -\frac{1}{3} \notin \mathbb{Z}$ $\frac{m+1}{n}+p = -\frac{1}{3}-\frac{5}{3} = -\frac{6}{3} = -2 \in \mathbb{Z}$

Prema pravilu III uvodimo smjenu $1+x^3 = x^3z^3$

$$1+x^3 = x^3 z^3$$
$$x^3 z^3 - x^3 = 1$$
$$(z^3-1)\,x^3 = 1$$
$$x^3 = \frac{1}{z^3-1}$$
$$x = \frac{1}{(z^3-1)^{\frac{1}{3}}}$$

$$1+x^3 = \frac{z^3}{z^3-1}$$
$$x^2\sqrt[3]{(1+x^3)^5} = \frac{1}{(z^3-1)^{\frac{2}{3}}}\sqrt[3]{\left(\frac{z^3}{z^3-1}\right)^5} =$$
$$= \frac{1}{(z^3-1)^{\frac{2}{3}}}\cdot\frac{z^5}{(z^3-1)^{\frac{5}{3}}} = \frac{z^5}{(z^3-1)^{\frac{7}{3}}}$$
$$dx = \left((z^3-1)^{-\frac{1}{3}}\right) dz = -\frac{1}{3}\cdot(z^3-1)^{-\frac{4}{3}}\cdot 3z^2\, dz =$$
$$= \frac{-z^2}{(z^3-1)^{\frac{4}{3}}}\,dz$$

$$\int\frac{dx}{x^2\sqrt[3]{(1+x^3)^5}} = \left|\begin{matrix} 1+x^3 = x^3z^3 \\ \vdots \end{matrix}\right| = \int\frac{-z^2}{(z^3-1)^{\frac{4}{3}}}\cdot\frac{(z^3-1)^{\frac{7}{3}}}{z^5}\,dz =$$
$$= -\int\frac{z^3-1}{z^3}\,dz = \int\frac{1-z^3}{z^3}\,dz = \int z^{-3}dz - \int dz = \frac{z^{-2}}{-2} - z + C$$
$$= -\frac{1}{2z^2} - z + C = \left|\begin{matrix} z^3 = x^{-3}+1 \\ z = \sqrt[3]{1+\frac{1}{x^3}}\end{matrix}\right| = -\frac{1}{2\sqrt[3]{\left(1+\frac{1}{x^3}\right)^2}} - \sqrt[3]{1+\frac{1}{x^3}} + C$$

e) $$\int\frac{2x^2-x-5}{\sqrt{x^2-2x}}\,dx$$

Prema pravilu $\overline{IV}$ (što je još poznato pod imenom metoda Ostrogradskog) imamo

$$I = \int\frac{2x^2-x-5}{\sqrt{x^2-2x}}\,dx = (Ax+B)\sqrt{x^2-2x} + D\int\frac{dx}{\sqrt{x^2-2x}} \quad \Big/\frac{d}{dx}$$

$$\frac{2x^2-x-5}{\sqrt{x^2-2x}} = A\sqrt{x^2-2x} + (Ax+B)\frac{2x-2}{2\sqrt{x^2-2x}} + \frac{D}{\sqrt{x^2-2x}} \quad \Big/\cdot\sqrt{x^2-2x}$$

$$2x^2-x-5 = A(x^2-2x) + (Ax+B)(x-1) + D$$
$$2x^2-x-5 = 2Ax^2 + (B-3A)x + (D-B)$$

Izjednačavamo koeficijente uz isti stepen

x^2: $2A = 2$ $\Rightarrow$ $A=1$

x: $B-3A = -1$ $B=2$

x^0: $D-B = -5$ $D=-3$

$$I = (x+2)\sqrt{x^2-2x} - 3\int\frac{dx}{\sqrt{x^2-2x}}$$

$$x^2-2x = x^2-2\cdot x\cdot 1+1-1$$
$$=(x-1)^2-1$$

$$\int\frac{dx}{\sqrt{x^2-2x}} = \int\frac{d(x-1)}{\sqrt{(x-1)^2-1}} = \ln\left|x-1-\sqrt{(x-1)^2-1}\right|$$

Prema tome
$$\int\frac{2x^2-x-5}{\sqrt{x^2-2x}}\,dx = (x+2)\sqrt{x^2-2x} - 3\ln\left|x-1-\sqrt{(x-1)^2-1}\right| + C$$

f) $$\int\frac{dx}{(x-1)\sqrt{1-x^2}}$$

Prema pravilu $\underline{\overline{V}}$ uvodimo smjenu $x-1=\frac{1}{t}$
$$dx = -\frac{1}{t^2}\,dt$$

$$I = \int\frac{dx}{(x-1)\sqrt{1-x^2}} = \left|\begin{matrix} x-1=\frac{1}{t} \\ dx = -\frac{1}{t^2}dt\end{matrix}\right| = \int\frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{1-\underbrace{\left(\frac{1}{t}+1\right)^2}_{\frac{1}{t^2}+\frac{2}{t}+1}}} =$$

$-1-2t>0$
$2t<-1$
$t<-\frac{1}{2}$
$|t|=-t$

$$= \int\frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{-\frac{1+2t}{t^2}}} = -\int\frac{\frac{dt}{t}}{\frac{\sqrt{-1-2t}}{|t|}} = -\int\frac{|t|\,dt}{t\sqrt{-1-2t}}$$

zato što je $\sqrt{t^2}=|t|$

$$= \int\frac{dt}{\sqrt{-1-2t}} = \int(-1-2t)^{-\frac{1}{2}}\left(-\frac{1}{2}\right)d(-1-2t) = -\frac{1}{2}\cdot\frac{(-1-2t)^{\frac{1}{2}}}{\frac{1}{2}} + C = C - \sqrt{-1-\frac{2}{x-1}}$$

Zadaci za vježbu

① $\displaystyle\int\frac{dx}{(1+\sqrt[3]{x})\sqrt{x}}$ ② $\displaystyle\int x\sqrt{3-x}\,dx$ ③ $\displaystyle\int\frac{1}{x}\sqrt{\frac{x-2}{x}}\,dx$

④ $\displaystyle\int\frac{dx}{\sqrt{(5-x^2)^3}}$ ⑤ $\displaystyle\int\frac{\sqrt{1+x^2}}{x^2}\,dx$ ⑥ $\displaystyle\int x^2\sqrt{4-x^2}\,dx$

⑦* $\displaystyle\int\frac{dx}{x\sqrt{x^2-9}}$ ⑧* $\displaystyle\int\frac{\sqrt[3]{(1+2x^3)^2}}{x^6}\,dx$

⑨ $\displaystyle\int\frac{dt}{t\sqrt{1-t^3}}$ ⑩ $\displaystyle\int\frac{x^2\,dx}{\sqrt{x^2+2x+3}}$

⑪ $\displaystyle\int\frac{x^2+4x}{\sqrt{x^2+2x+2}}\,dx$ ⑫* $\displaystyle\int\frac{x^2\,dx}{\sqrt{2ax-x^2}}$

Rješenja:

1. $6\sqrt[6]{x}-6\operatorname{arctg}\sqrt[6]{x}$ 2. $0{,}4(x^3-x-6)\sqrt{3-x}$

3. $-2\sqrt{\frac{x-2}{\cancel{x}}}-\ln\left[|x|\left(1-\sqrt{\frac{x-2}{x}}\right)^2\right]$ 4. $\dfrac{x}{5\sqrt{5-x^2}}$ 5. $\ln(x+\sqrt{x^2+1})$
$-\dfrac{\sqrt{x^2+1}}{x}$ 6. $2\arcsin\frac{x}{2}+\frac{x}{4}(x^2-2)\sqrt{4-x^2}$ 7. $\pm\frac{1}{3}\arccos\frac{3}{x}$
gdje je + kad $x>0$, − kad $x<0$. 8. $-\frac{1}{5}x^{-5}(2x^3+1)^{\frac{5}{3}}$

9. $\dfrac{1}{3}\ln\dfrac{|1-\sqrt{1-x^3}|}{1+\sqrt{1-x^3}}$ 10. $\dfrac{x-3}{2}\sqrt{x^2+2x+3}$ 11. $\dfrac{x+5}{2}\sqrt{x^2+2x+2}-$
$-\frac{7}{2}\ln(x+1+\sqrt{x^2+2x+2})$ 12. $\dfrac{3a^2}{2}\arcsin\dfrac{x-a}{a}-\dfrac{x+3a}{2}\sqrt{2ax-x^2}$

Izabrani **Zadaci za vježbu** sa rješenjima

(iz lekcije Integracija nekih iracionalnih funkcija)

Metoda Ostrogradskog

$$\int\frac{P_n(x)}{\sqrt{ax^2+bx+c}}\,dx=Q_{n-1}(x)\cdot\sqrt{ax^2+bx+c}+\lambda\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

① $\displaystyle I=\int\frac{3x^3}{\sqrt{x^2+4x+5}}\,dx$ Rj.

$$=(ax^2+bx+c)\sqrt{x^2+4x+5}+\lambda\int\frac{dx}{\sqrt{x^2+4x+5}}\quad\Big/\frac{d}{dx}$$

$$\frac{3x^3}{\sqrt{x^2+4x+5}}=(2ax+b)\sqrt{x^2+4x+5}+(ax^2+bx+c)\cdot\frac{2x+4}{2\sqrt{x^2+4x+5}}+\lambda\cdot\frac{1}{\sqrt{x^2+4x+5}}\quad\Big/\cdot\sqrt{x^2+4x+5}$$

$$3x^3=(2ax+b)(x^2+4x+5)+(ax^2+bx+c)(x+2)+\lambda$$

$$3x^3=(\underline{2ax^3}+\underline{\underline{8ax^2}}+\underline{\underline{\underline{10ax}}})+b(\underline{\underline{x^2}}+\underline{\underline{\underline{4x}}}+5)+(\underline{ax^3}+\underline{\underline{bx^2}}+\underline{\underline{\underline{cx}}})+\underline{\underline{2ax^2}}+\underline{\underline{\underline{2bx}}}+2c+\lambda$$

$$3x^3=(2a+a)x^3+(8a+b+b+2a)x^2+(10a+4b+c+2b)x+5b+2c+\lambda$$

$$3x^3=3ax^3+(10a+2b)x^2+(10a+6b+c)x+5b+2c+\lambda$$

uz x^3: $3a=3$ $\Rightarrow$ $a=1$
uz x^2: $10a+2b=0$ $\Rightarrow$ $b=-5$
uz x: $10a+6b+c=0$ $\Rightarrow$ $c=20$
uz x^0: $\underline{5b+2c+\lambda=0}$ $\lambda=-15$

$$I=(x^2-5x+20)\sqrt{x^2+4x+5}-15\underbrace{\int\frac{dx}{\sqrt{x^2+4x+5}}}_{I_1}$$

$x^2+4x+5=x^2+2\cdot x\cdot 2+2^2-2^2+5=(x+2)^2+1$, $\displaystyle I_1=\int\frac{dx}{\sqrt{(x+2)^2+1}}=\ln\left|x+2+\sqrt{(x+2)^2+1}\right|+c$

$$I=(x^2-5x+20)\sqrt{x^2+4x+5}-15\ln\left|x+2+\sqrt{(x+2)^2+1}\right|+c$$

② $$I=\int\frac{3x+1}{\sqrt{2x^2-x+1}}\,dx \overset{Rj.}{=} a\sqrt{2x^2-x+1}+\lambda\int\frac{dx}{\sqrt{2x^2-x+1}} \quad \Big/ \frac{d}{dx}$$

$$\frac{3x+1}{\sqrt{2x^2-x+1}}=a\cdot\frac{4x-1}{2\sqrt{2x^2-x+1}}+\lambda\cdot\frac{1}{\sqrt{2x^2-x+1}} \quad \Big/\cdot 2\sqrt{2x^2-x+1}$$

$$6x+2=a(4x-1)+2\lambda \Rightarrow \begin{array}{l} 4a=6 \\ \underline{2\lambda-a=2} \end{array} \qquad \begin{array}{l} a=\frac{6}{4}=\frac{3}{2} \\ \lambda=\frac{7}{4} \end{array}$$

$$I=\frac{3}{2}\sqrt{2x^2-x+1}+\frac{7}{4}\int\frac{dx}{\sqrt{2x^2-x+1}}=\frac{3}{2}\sqrt{2x^2-x+1}+\frac{7}{4}I_1$$

$$2x^2-x+1=2\left(x^2-\frac{1}{2}x+\frac{1}{2}\right)=2\left(x^2-2\cdot x\cdot\frac{1}{4}+\left(\frac{1}{4}\right)^2-\left(\frac{1}{4}\right)^2+\frac{1}{2}\right)=2\left[\left(x-\frac{1}{4}\right)^2+\frac{7}{16}\right]$$

$$I_1=\int\frac{dx}{\sqrt{2\left[\left(x-\frac{1}{4}\right)^2+\frac{7}{16}\right]}}=\frac{1}{\sqrt{2}}\int\frac{dx}{\sqrt{\left(x-\frac{1}{4}\right)^2+\frac{7}{16}}}=\left|\begin{array}{l} x-\frac{1}{4}=\frac{\sqrt{7}}{4}t \quad /\cdot 4 \\ dx=\frac{\sqrt{7}}{4}dt \\ 4x-1=\sqrt{7}t \end{array}\right|=\frac{1}{\sqrt{2}}\cdot\int\frac{\frac{\sqrt{7}}{4}dt}{\sqrt{\frac{7}{16}t^2-\frac{7}{16}}}$$

$$=\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{7}}{4}\cdot\frac{4}{\sqrt{7}}\int\frac{dt}{\sqrt{t^2-1}}=\frac{1}{\sqrt{2}}\ln\left|t+\sqrt{t^2+1}\right|+C=\frac{1}{\sqrt{2}}\ln\left|\frac{4x-1}{\sqrt{7}}+\sqrt{\left(\frac{4x-1}{\sqrt{7}}\right)^2+1}\right|+C$$

$$I=\frac{3}{2}\sqrt{2x^2-x+1}+\frac{7}{4\sqrt{2}}\ln\left|\frac{4x-1}{\sqrt{7}}+\sqrt{\left(\frac{4x-1}{\sqrt{7}}\right)^2+1}\right|+C$$

Metodom Ostrogradskog rješavamo i integrale oblika

$$\int\sqrt{ax^2+bx+c}\,dx=\int\frac{ax^2+bx+c}{\sqrt{ax^2+bx+c}}\,dx \; .$$

③ $$\int\sqrt{x^2+1}\,dx \overset{Rj.}{=} (ax+b)\sqrt{x^2+1}+\lambda\int\frac{dx}{\sqrt{x^2+1}} \quad \Big/\frac{d}{dx}$$

$$\sqrt{x^2+1}=a\sqrt{x^2+1}+(ax+b)\frac{2x}{2\sqrt{x^2+1}}+\lambda\cdot\frac{1}{\sqrt{x^2+1}} \quad \Big/\cdot\sqrt{x^2+1}$$

$$x^2+1=\underline{a(x^2+1)}+\underline{(ax^2+bx)}+\underline{\lambda}$$

x^2: $a+a=1$, $2a=1$, $a=\frac{1}{2}$ $\qquad$ x: $b=0$ $\qquad$ x^0: $a+\lambda=1$, $\lambda=1-\frac{1}{2}$, $\lambda=\frac{1}{2}$

$$\int\sqrt{x^2+1}\,dx=\frac{1}{2}x\sqrt{x^2+1}+\frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right|+C$$

④$^{\vee}$ $$\int\frac{2x^2-3x}{\sqrt{x^2-2x+5}}\,dx$$

⑤$^{\vee}$ $$\int\sqrt{x^2-2x-1}\,dx$$

⑥$^{\vee}$ $$\int\frac{x^5}{\sqrt{1-x^2}}\,dx \qquad Rj. \; -\frac{8+4x^2+3x^4}{15}\sqrt{1-x^2}$$

⑦ $$\int x^4\sqrt{1-x^2}\,dx \qquad \text{uputa: } \int\frac{x^4(1-x^2)}{\sqrt{1-x^2}}\,dx$$

$$\int R\left(x,\sqrt[n]{ax+b}\right)dx\,, \qquad \text{smjena } \; ax+b=t^n$$

① $$\int\frac{dx}{\sqrt{2x-1}-\sqrt[4]{2x-1}}=\left|\begin{array}{l} 2x-1=t^4 \\ 2dx=4t^3dt \quad /:2 \\ dx=2t^3dt \\ t=\sqrt[4]{2x-1} \end{array}\right|=2\int\frac{t^3\,dt}{\sqrt{t^4}-\sqrt[4]{t^4}}=2\int\frac{t^3}{\underbrace{t^2-t}_{t(t-1)}}dt$$

$$=2\int\frac{t^2}{t-1}dt=2\int\frac{t^2-1+1}{t-1}dt=2\int\frac{t^2-1}{t-1}dt+\int\frac{dt}{t-1}=$$

$$=2\int\frac{(t-1)(t+1)}{t-1}dt+2\int\frac{dt}{t-1}=2\int(t-1)\,dt+2\int\frac{1}{t-1}dt=2\cdot\frac{t^2}{2}-2t+2\ln|t-1|+C$$

$$=\sqrt[4]{(2x-1)^2}+2\sqrt[4]{(2x-1)}+2\ln\left|\sqrt[4]{2x-1}-1\right|+C=\sqrt{2x-1}+2\sqrt[4]{2x-1}+2\ln\left|\sqrt[4]{2x-1}-1\right|+C$$

② $$\int\frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}\,dx=\left|\begin{array}{l} x=t^4 \\ dx=4t^3dt \\ t=\sqrt[4]{x} \end{array}\right|=\int\frac{\sqrt[3]{1+t}}{t^2}\cdot 4t^3\,dt=$$

$$=4\int t\sqrt[3]{1+t}\,dt=\left|\begin{array}{l} 1+t=u^3 \\ dt=3u^2du \\ u=\sqrt[3]{1+t} \\ t=u^3-1 \end{array}\right|=4\int(u^3-1)\sqrt[3]{u^3}\cdot 3u^2\,du=$$

$$=12\int(u^6-u^3)\,du=12\left(\frac{u^7}{7}-\frac{u^4}{4}\right)+C=\frac{12}{7}u^7-3u^4+C=$$

$$= \frac{12}{7}\sqrt[3]{(1+t)^7} - 3\sqrt[3]{(1+t)^4} + c = \frac{12}{7}\sqrt[3]{(1+\sqrt[4]{x})^7} - 3\sqrt[3]{(1+\sqrt[4]{x})^4} + C$$

③ $\int\sqrt{\frac{x+1}{x-1}}\,dx$ ako stavim smjenu $\frac{x+1}{x-1}=t^2$ dobiću

$x+1=t^2(x-1)$

$x+1=t^2x-t^2$

$x-t^2x=-t^2-1$

$(1-t^2)x=-t^2-1 \quad /(-1)$

$x=\frac{t^2+1}{t^2-1}$

$dx=d\left(\frac{t^2+1}{t^2-1}\right)$

$dx=\frac{2t(t^2-1)-(t^2+1)\cdot 2t}{(t^2-1)^2}dt$

$dx=\frac{2t^3-2t-2t^3-2t}{(t^2-1)^2}dt$

$$I=\int t\cdot\frac{-4t}{(t^2-1)^2}dt=-4\int t\cdot\frac{t}{(t^2-1)^2}dt=-4I_1 \qquad dx=\frac{-4t}{(t^2-1)^2}dt$$

$$I_1=\int t\cdot\frac{t}{(t^2-1)^2}dt=\left|\begin{array}{ll} u=t & dv=\frac{t}{(t^2-1)^2}dt \\ du=dt & v=\int\frac{t}{(t^2-1)^2}dt=\left|\begin{array}{l} t^2-1=z \\ 2tdt=dz \\ tdt=\frac{dz}{2}\end{array}\right|=\int\frac{\frac{1}{2}dz}{z^2}=\end{array}\right.$$

$$=\frac{1}{2}\int z^{-2}dz=\frac{1}{2}\cdot\frac{z^{-1}}{-1}=-\frac{1}{2}\cdot\frac{1}{z}=-\frac{1}{2(t^2-1)}\quad\Big|$$

$$=-\frac{1}{2}\cdot\frac{t}{t^2-1}+\frac{1}{2}\int\frac{dt}{t^2-1}=-\frac{1}{2}\cdot\frac{t}{t^2-1}+\frac{1}{2}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+c_1$$

$$I=-4\cdot\left(-\frac{1}{2}\cdot\frac{t}{t^2-1}+\frac{1}{4}\ln\left|\frac{t-1}{t+1}\right|\right)+c=$$

$$=2\cdot\frac{\sqrt{\frac{x+1}{x-1}}}{\frac{x+1}{x-1}-1}-1\ln\left|\frac{\sqrt{\frac{x+1}{x-1}}-1}{\sqrt{\frac{x+1}{x-1}}+1}\right|+C$$

④^{v} $\int\frac{dx}{\sqrt{x}+\sqrt[3]{x}}$ Rj.: $6\sqrt[6]{x}-3\sqrt[3]{x}+2\sqrt{x}-6\ln(1+\sqrt[6]{x})+C$

⑤^{v} $\int\frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}dx$ Rj.: $\ln\left|\frac{(\sqrt{x+1}-1)^2}{x+2+\sqrt{x+1}}\right|-\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2\sqrt{x+1}+1}{\sqrt{3}}+C$

⑥^{v} $\int\frac{dx}{(2-x)\sqrt{1-x}}$ Rj. $-2\operatorname{arctg}\sqrt{1-x}+c$

integrali koji se mogu riješiti racionalisanjem

① $$I=\int\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}dx$$

Rj.

$$I=\int\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\cdot\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}dx=\int\frac{x+1-2\cdot\sqrt{x+1}\cdot\sqrt{x-1}+x-1}{x+1-(x-1)}dx=$$

$$=\frac{1}{2}\int(2x-2\sqrt{x^2-1})dx=\int x\,dx-\int\sqrt{x^2-1}\,dx=\frac{x^2}{2}-I_1$$

$$I_1=\int\sqrt{x^2-1}\,dx=(ax+b)\sqrt{x^2-1}+\lambda\int\frac{dx}{\sqrt{x^2-1}}\quad\Big/\frac{d}{dx}$$

$$\sqrt{x^2-1}=a\cdot\sqrt{x^2-1}+(ax+b)\frac{2x}{2\sqrt{x^2-1}}+\lambda\cdot\frac{1}{\sqrt{x^2-1}}\quad\Big/\sqrt{x^2-1}$$

$$x^2-1=\underline{a(}\underline{\underline{x^2}}-1)+(\underline{ax^2}+\underline{\underline{bx}})+\lambda$$

x^2: $a+a=1 \Rightarrow a=\frac{1}{2}$, x: $b=0$, x^0: $-a+\lambda=-1$ $\lambda=-\frac{1}{2}$

$$I_1=\int\sqrt{x^2-1}\,dx=\frac{1}{2}\sqrt{x^2-1}-\frac{1}{2}\ln|x+\sqrt{x^2-1}|+c$$

$$I=\frac{1}{2}x^2-\frac{1}{2}\sqrt{x^2-1}+\frac{1}{2}\ln|x+\sqrt{x^2-1}|+c$$

② $$\int\frac{x}{\sqrt{x+2}+\sqrt{x+1}}dx$$

Rj.

$$\int\frac{x}{\sqrt{x+2}+\sqrt{x+1}}\cdot\frac{\sqrt{x+2}-\sqrt{x+1}}{\sqrt{x+2}-\sqrt{x+1}}dx=\int\frac{x\sqrt{x+2}-x\sqrt{x+1}}{x+2-(x+1)}dx=$$

$$=\int(x\sqrt{x+2}-x\sqrt{x+1})dx=\int x\sqrt{x+2}\,dx-\int x\sqrt{x+1}\,dx=I_1-I_2$$

$$I_1=\int x\sqrt{x+2}\,dx=\left|\begin{array}{l}x+2=t^2\\ dx=2t\,dt\\ x=t^2-2\\ t=\sqrt{x+2}\end{array}\right|=\int(t^2-2)\,t\cdot 2t\,dt=2\int(t^4-2t^2)\,dt=2\cdot\frac{t^5}{5}-4\frac{t^3}{3}+C_1$$

$$=\frac{2}{5}\sqrt{(x+2)^5}-\frac{4}{3}\sqrt{(x+2)^3}+C_1$$

$$I_2=\int x\sqrt{x+1}\,dx=\left|\begin{array}{l}x+1=t^2\\ x=t^2-1\\ dx=2t\,dt\\ t=\sqrt{x+1}\end{array}\right|=\int(t^2-1)\cdot t\cdot 2t\,dt=2\int(t^4-t^2)\,dt=$$

$$=2\frac{t^5}{5}-2\cdot\frac{t^3}{3}+C_2=\frac{2}{5}\sqrt{(x+1)^5}-\frac{2}{3}\sqrt{(x+1)^3}+C_2$$

$$I=\frac{2}{5}\sqrt{(x+2)^5}-\frac{4}{3}\sqrt{(x+2)^3}-\frac{2}{5}\sqrt{(x+1)^5}+\frac{2}{3}\sqrt{(x+1)^3}+e$$

③ ✓ $\int\frac{dx}{x-\sqrt{x^2-1}}$. Rj: $\frac{1}{2}x^2+\frac{1}{2}x\sqrt{x^2-1}-\frac{1}{2}\ln|x+\sqrt{x^2-1}|+e$

④ ✓ $\int\frac{dx}{\sqrt{x^2+1}-x}$

⑤ ✓ $\int\frac{\sqrt{x^2+2x+2}}{x}\,dx$

$$\int\frac{Mx+N}{(x-\alpha)^n\sqrt{ax^2+bx+c}}\,dx,\quad n\in\mathbb{N},\quad M,N,a,b,c\in\mathbb{R}$$

uvodimo smjenu $x-\alpha=\frac{1}{t}$

① $$\int\frac{dx}{(x+1)\sqrt{x^2+x+1}}=\left|\begin{array}{l}x+1=\frac{1}{t}\\ dx=-\frac{1}{t^2}dt\\ t=\frac{1}{x+1}\end{array}\right|=\int\frac{-\frac{1}{t^2}dt}{t\sqrt{\underbrace{\left(\frac{1}{t}-1\right)^2+\frac{1}{t}-1+1}_{\frac{1}{t^2}-\frac{2}{t}+1+\frac{1}{t}}}}=-\int\frac{dt}{t\sqrt{\frac{1}{t^2}-\frac{1}{t}+1}}$$

$$=-\int\frac{dt}{\sqrt{t^2-t+1}}=\left|\begin{array}{l}t^2-t+1=\\ =t^2-2\cdot t\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1\\ \quad=\left(t-\frac{1}{2}\right)^2+\frac{3}{4}\end{array}\right|=-\int\frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}}}=\left|\begin{array}{l}t-\frac{1}{2}=\frac{\sqrt{3}}{2}z\\ dt=\frac{\sqrt{3}}{2}dz\\ \sqrt{3}z=2t-1\end{array}\right|$$

$$=-\frac{\sqrt{3}}{2}\int\frac{dz}{\sqrt{\frac{3}{4}z^2+\frac{3}{4}}}=-\frac{\sqrt{3}}{2}\cdot\frac{2}{\sqrt{3}}\int\frac{dz}{\sqrt{z^2+1}}=-\ln|z+\sqrt{z^2+1}|+C=$$

$$=-\ln\left|\frac{2t-1}{\sqrt{3}}+\sqrt{\left(\frac{2t-1}{\sqrt{3}}\right)^2+1}\right|+C=-\ln\left|\frac{\frac{2}{x+1}-1}{\sqrt{3}}+\sqrt{\frac{\left(\frac{2}{x+1}-1\right)^2}{3}+1}\right|+C$$

② $$I=\int\frac{dx}{(x-1)^3\sqrt{x^2+3x+1}}=\left|\begin{array}{l}x-1=\frac{1}{t}\\ dx=-\frac{1}{t^2}dt\\ x=\frac{1}{t}+1\\ t=\frac{1}{x-1}\end{array}\right|=\int\frac{-\frac{1}{t^2}dt}{\frac{1}{t^3}\sqrt{\underbrace{\left(\frac{1}{t}+1\right)^2+3\left(\frac{1}{t}+1\right)+1}_{\frac{1}{t^2}+\frac{2}{t}+1+\frac{3}{t}+3+1}}}=-\int\frac{t\,dt}{\sqrt{\frac{1}{t^2}+\frac{5}{t}+1}}$$

$$=-\int\frac{t\,dt}{\sqrt{\frac{1+5t+5t^2}{t^2}}}=-\int\frac{t^2}{\sqrt{5t^2+5t+1}}dt=(at+b)\sqrt{5t^2+5t+1}+\lambda\int\frac{dt}{\sqrt{5t^2+5t+1}}\ \Big/\cdot\frac{d}{dt}$$

$$\frac{-t^2}{\sqrt{5t^2+5t+1}}=a\sqrt{5t^2+5t+1}+(at+b)\frac{10t+5}{2\sqrt{5t^2+5t+1}}+\lambda\cdot\frac{1}{\sqrt{5t^2+5t+1}}\ \Big/\cdot 2\sqrt{5t^2+5t+1}$$

$$-2t^2=2a\cdot(\underline{5t^2}+\underline{\underline{5t}}+\underline{\underline{\underline{1}}})+a(\underline{10t^2}+\underline{\underline{5t}})+b(\underline{\underline{10t}}+5)+2\lambda$$

t^2: $10a+10a=-2$

$a=-\frac{1}{10}$

t: $10a+5a+10b=0$

$15a+10b=0$

$10b=\frac{15}{10}$

$10b=\frac{3}{2}$

$b=\frac{3}{20}$

t^0: $2a+5b+2\lambda=0$

$-\frac{2}{10}+\frac{15}{20}=-2\lambda$

$-2\lambda=\frac{11}{20}$

$\lambda=-\frac{11}{40}$

$$I=\left(-\frac{1}{10}t+\frac{3}{20}\right)\sqrt{5t^2+5t+1}-\frac{11}{40}\int\frac{dt}{\sqrt{5t^2+5t+1}}$$ SAMI ZAVRŠITI

③ ✓ $\int\frac{dx}{x^2\sqrt{x^2-x+1}}$

④ ✓ $\int\frac{(3x+2)\,dx}{(x+1)\sqrt{x^2+3x+3}}$

⑤ ✓ $\int\frac{dx}{x^3\sqrt{x^2+1}}$

integral binomnog diferencijala

$$\int x^m(a+bx^n)^p\,dx\quad(a,b\in\mathbb{R};\ m,n,p\in\mathbb{Q})$$

Integracija je moguća ako

1° $p\in\mathbb{Z}$, uvodimo smjenu $x=t^s$, $s=NZS(m_2,n_2)$, $m=\frac{m_1}{m_2}$, $n=\frac{n_1}{n_2}$

nazivnika od m i n

2° $\frac{m+1}{n}\in\mathbb{Z}$, uvodimo smjenu $a+bx^n=t^{p_2}$, $p=\frac{p_1}{p_2}$

3° $\frac{m+1}{n}+p\in\mathbb{Z}$, uvodimo smjenu $ax^{-n}+b=t^{p_2}$, p_2 nazivnik od p

integral binomnog diferencijala

$\int x^m (a+bx^n)^p \, dx \quad (a,b \in \mathbb{R};\ m,n,p \in \mathbb{Q})$

Integracija je moguća ako

1° $p \in \mathbb{Z}$, uvodimo smjenu $x = t^s$, $s = NZS(m_2, n_2)$, $m = \frac{m_1}{m_2}$, $n = \frac{n_1}{n_2}$

nazivnika od m i n

2° $\frac{m+1}{n} \in \mathbb{Z}$, uvodimo smjenu $a+bx^n = t^{p_2}$, $p = \frac{p_1}{p_2}$

3° $\frac{m+1}{n} + p \in \mathbb{Z}$, uvodimo smjenu $ax^{-n}+b = t^{p_2}$, p_2 nazivnik od p

①
$$\int \frac{dx}{x^2(\sqrt{1+x^2})^3} = \int x^{-2}(1+x^2)^{-\frac{3}{2}} dx = \left| \begin{array}{l} m=-2,\ n=2,\ p=-\frac{3}{2} \\ p \notin \mathbb{Z}, \text{ nije } 1° \\ \frac{m+1}{n} = \frac{-2+1}{2} = -\frac{1}{2} \notin \mathbb{Z}, \text{ nije } 2° \\ \frac{m+1}{n} + p = -\frac{1}{2} - \frac{3}{2} = -2 \in \mathbb{Z},\ 3° \end{array} \right.$$

smjena: $x^{-2}+1 = t^2$

$x^{-2} = t^2 - 1$

$x^2 = (t^2-1)^{-1}$

$x = (t^2-1)^{-\frac{1}{2}}$

$\searrow$ $1+x^2 = 1 + (t^2-1)^{-1}$

$dx = -\frac{1}{2}(t^2-1)^{-\frac{3}{2}} \cdot 2t\,dt$

$dx = -t(t^2-1)^{-\frac{3}{2}} dt$

$$= \int (t^2-1)\left(1+\frac{1}{t^2-1}\right)^{-\frac{3}{2}} \cdot (-t)(t^2-1)^{-\frac{3}{2}} dt$$

$$= \int (t^2-1) \cdot \frac{t^{-3}}{(t^2-1)^{-\frac{3}{2}}} (-t) \cdot (t^2-1)^{-\frac{3}{2}} dt$$

$$= \int (-1+t^{-2})\, dt = -t - \frac{1}{t} + c = \frac{-x^{-2}-2}{\sqrt{x^{-2}+1}} + C$$

Eulerove smjene $\int R(x, \sqrt{ax^2+bx+c})\, dx$ R-racionalna f-ja

1° za $a>0$ uzimamo smjenu $\sqrt{ax^2+bx+c} = \pm\sqrt{a}\,x + t$

2° za $c>0$ uzimamo smjenu $\sqrt{ax^2+bx+c} = xt \pm \sqrt{c}$

3° za $b^2-4ac>0$ uzimamo smjenu $\sqrt{a(x-x_1)(x-x_2)} = t(x-x_1)$

$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$

① $I = \int \frac{x - \sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}$ Rj. 3°

smjena $\sqrt{(x-1)(x+2)} = t(x+1)$

Integral binomnog diferencijala

$\int x^m (a+bx^n)^p dx \quad (m, n, p \in \mathbb{Q})$

Integracija je moguća:

1° $p \in \mathbb{Z}$

smjena: $x = t^s$

s = NZS nazivnika od m i n

2° $\frac{m+1}{n} \in \mathbb{Z}$

$a + bx^n = t^s$ — smjena

s - nazivnik od p

3° $\frac{m+1}{n} + p \in \mathbb{Z}$

$ax^{-n} + b = t^s$

s - nazivnik od p

$\sqrt[n]{x^m} = x^{\frac{m}{n}}$

1. $\int x^{-\frac{3}{4}} \cdot (1 + x^{\frac{1}{6}})^{-1} dx =$

$$\left| \begin{array}{l} m = -\frac{3}{4},\ n = \frac{1}{6},\ p = -1 \\ x = t^{12} \Rightarrow dx = 12t^{11} dt \end{array} \right| =$$

$$= \int (t^{12})^{-\frac{3}{4}} \cdot \left(1 + (t^{12})^{\frac{1}{6}}\right)^{-1} \cdot 12t^{11} dt =$$

$$= 12\int t^{-9} \cdot (1+t^2)^{-1} \cdot t^{11} dt = 12\int \frac{t^2+1-1}{1+t^2} dt =$$

$$= 12\int \left(1 - \frac{1}{t^2+1}\right) dt = 12 \cdot \int dt - \int \frac{1}{t^2+1} = 12t - \text{arctg}\,t + C$$

$$= 12\sqrt[12]{x} - \text{arctg}\sqrt[12]{x} + C$$

2. $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}dx = \int x^{-\frac{2}{3}}\cdot\left(1+x^{\frac{1}{3}}\right)^{\frac{1}{2}}dx =$

$m=\frac{-2}{3}$, $n=\frac{1}{3}$, $p=\frac{1}{2}$

$\frac{m+1}{n} = \frac{\frac{1}{3}}{1} = 1$

2° slučaj

$1+x^{\frac{1}{3}} = t^2$

$\frac{1}{3}x^{\frac{1}{3}-1}dx = 2t\,dt \;/\cdot 3$

$x^{-\frac{2}{3}}dx = 6t\,dt$

$= 6\int (t^2)^{\frac{1}{2}}\cdot t\,dt = 6\int t^2\,dt = \overset{2}{\cancel{6}}\cdot\frac{t^3}{\cancel{3}} + C = 2t^3 + C$

$= 2\cdot\left(\sqrt{1+\sqrt[3]{x}}\right)^3 + C$

3. $\int \frac{dx}{x^2\cdot\left(\sqrt{1+x^2}\right)^3} = \int x^{-2}(1+x^2)^{-\frac{3}{2}}dx =$

$m=-2$, $n=2$, $p=-\frac{3}{2}$

$\frac{m+1}{n}+p = -\frac{1}{2}-\frac{3}{2} = -2$

3° slučaj

smjena: $x^{-2}+1 = t^2$ $\qquad dx = -\frac{1}{2}\cdot(t^2-1)^{-\frac{3}{2}}\cdot 2t\,dt$

$x^{-2} = t^2-1$ $\qquad dx = -t\cdot(t^2-1)^{-\frac{3}{2}}dt$

$x^2 = (t^2-1)^{-1}$

$x = (t^2-1)^{-\frac{1}{2}}$

$= \int (t^2-1)\cdot\left(1+\frac{1}{t^2-1}\right)^{-\frac{3}{2}}\cdot(-t)\cdot(t^2-1)^{-\frac{3}{2}}dt =$

$= \int (t^2-1)\cdot\left(\frac{t^2-1+1}{t^2+1}\right)^{-\frac{3}{2}}\cdot(-t)\cdot(t^2-1)^{-\frac{3}{2}}dt =$

$\int (t^2-1)\cdot\frac{t^{-3}}{\cancel{(t^2-1)^{-\frac{3}{2}}}}\cdot(-t)\cdot\cancel{(t^2-1)^{-\frac{3}{2}}}dt$

$= \int(-1 + t^{-2})dt = -t - \frac{1}{t} + C = \frac{-t^2-1}{t} + C =$

$= \frac{-x^{-2}-1-1}{\sqrt{x^{-2}+1}} + C = \frac{-x^{-2}-2}{\sqrt{x^{-2}+1}} + C$

za vježbu:

④ $\int \frac{dx}{\sqrt[3]{(1+x^2)^2}}$ ([illegible]) ✓

⑤ $\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}dx$ – I tip ✓

⑥ $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx$ – II tip ✓

⑦ $\int \frac{dx}{x^3\cdot\sqrt[3]{2-x^3}}$ – III tip ✓

Eulerove smjene

$I = \int R(x, \sqrt{ax^2+bx+c})\,dx$

R - racionalna funkcija

1° $\sqrt{ax^2+bx+c} = \pm\sqrt{a}\cdot x + t$ (ako je $a>0$)

2° $\sqrt{ax^2+bx+c} = xt \pm \sqrt{c}$ (ako je $c>0$)

3° $\sqrt{ax^2+bx+c} = t\cdot(x-x_1)$ (ako je $ax^2+bx+c = a\cdot(x-x_1)(x-x_2$

$x_1, x_2 \in R$)

1. $I = \int \frac{dx}{x+\sqrt{x^2+x+1}}$

smjena : $\sqrt{x^2+x+1} = -x+t \quad /^2$

$\cancel{x^2}+x+1 = \cancel{x^2}-2xt+t^2$

$x+2xt = t^2-1$

$x\cdot(1+2t) = t^2-1$

$x = \frac{t^2-1}{1+2t}$

$dx = \frac{2t\cdot(1+2t)-2\cdot(t^2-1)}{(1+2t)^2}$

$dx = \frac{2t+4t^2-2t^2+2}{(1+2t)^2}dt$

$dx = \frac{2t^2+2t+2}{(1+2t)^2}dt$

$I = \int \frac{\frac{2t^2+2t+2}{(1+2t)^2\cancel{dt}}}{t}dt =$

$= \int \frac{2t^2+2t+2}{t\cdot(1+2t)^2}dt$

$\frac{2t^2+2t+2}{t\cdot(1+2t)^2} = \frac{A}{t} + \frac{B}{(1+2t)} + \frac{C}{(1+2t)^2} \quad /\cdot t\cdot(1+2t)^2$

$2t^2+2t+2 = A\cdot(1+2t)^2 + Bt(1+2t) + Ct$

$A = 2, \quad B = -3, \quad C = -3$

$I = \int\frac{2}{t}dt - 3\int\frac{1}{1+2t}dt - 3\int\frac{1}{(1+2t)^2}dt =$

$= 2\ln|t| - \frac{3}{2}\ln|1+2t| + \frac{3}{2(1+2t)} + C$

$t = x+\sqrt{x^2+x+1}$

$I = \frac{dx}{1+\sqrt{1-2x-x^2}}$

$c = 1$

$xt-1 = \sqrt{1-2x-x^2} \quad /^2$

$x^2t^2-2xt+\cancel{1} = \cancel{1}-2x-x^2 \quad /:x$

$xt^2-2t = -2-x$

$xt^2+x = 2t-2$

$x\cdot(t^2+1) = 2t-2$

$x = \frac{2t-2}{t^2+1}$

$dx = \frac{2(t^2+1)-2t(2t-2)}{(t^2+1)^2}dt$

$dx = \frac{2t^2+2-4t^2+4t}{(t^2+1)^2}dt$

$dx = \frac{-2t^2+4t+2}{(t^2+1)^2}dt$

$I = \int\frac{\frac{-2t^2+4t+2}{(t^2+1)^2}}{\frac{2t-2}{t^2+1}\cdot t}dt = \int\frac{\cancel{2}\cdot(-t^2+2t+1)\,dt}{\cancel{2}\cdot(t-1)\cdot t\cdot(t^2+1)}$

$\frac{-t^2+2t+1}{t\cdot(t-1)(t^2+1)} = \frac{A}{t} + \frac{B}{t-1} + \frac{Ct+D}{t^2+1}$ ...... A, B, C, D za vj.

$A = -1, \quad B = 1, \quad C = 0, \quad D = 2$

$= -\int\frac{1}{t}dt + \int\frac{1}{t-1}dt + \int\frac{2\,dt}{t^2+1} =$

$= -\ln|t| + \ln|t-1| + 2\,\mathrm{arctg}\,t + C$

$= \ln\left|\frac{t-1}{t}\right| + 2\,\mathrm{arctg}\,t + C$

$t = \frac{1+\sqrt{1-2x+x^2}}{x}$

3. $$I=\int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,dx$$

$$x^2+3x+2=(x+1)\cdot(x+2)$$

$$\sqrt{x^2+3x+2}=t\,(x+1)\ \Big/^{2}$$

$$x^2+3x+2=t^2(x+1)^2$$

$$(x+1)(x+2)=t^2(x+1)^2\ \Big/:(x+1)$$

$$x+2=t^2\cdot(x+1)$$

$$x+2=t^2x+t^2$$

$$x-t^2x=t^2-2$$

$$x(1-t^2)=t^2-2$$

$$x=\frac{t^2-2}{1-t^2}$$

$$dx=\frac{2t\cdot(1-t^2)+2t\cdot(t^2-2)}{(1-t^2)^2}\,dt$$

$$dx=\frac{2t-2t^3+2t^3-4t}{(1-t^2)^2}\,dt$$

$$dx=\frac{-2t}{(1-t^2)^2}\,dt$$

$$I=\int \frac{\frac{t^2-2}{1-t^2}+\frac{t}{1-t^2}}{\frac{t^2-2}{1-t^2}-\frac{t}{1-t^2}}\cdot\frac{-2t}{(1-t^2)^2}\,dt=$$

$$\left[\sqrt{x^2+3x+2}=t\cdot\left(\frac{t^2-2}{1-t^2}+1\right)=t\cdot\frac{t^2-2+1-t^2}{1-t^2}=\frac{-t}{1-t^2}\right]$$

$$=\int\frac{(t^2+t-2)\cdot(-2t)}{(t^2-t-2)(1-t^2)^2}\,dt$$

$$=-2\int\frac{(t-1)(t+2)\,t}{(t+1)\cdot(t-2)\cdot(t-1)^2(t+1)^2}\,dt$$

$$-2\int\frac{(t^2+2t)}{(t-1)\cdot(t-2)\cdot(t+1)^3}\,dt\ ,$$

$$=\frac{-2t^2-4t}{(t-1)\cdot(t-2)\,(t+1)^3}=\frac{A}{t-1}+\frac{B}{t-2}+\frac{C}{t+1}+\frac{D}{(t+1)^2}+\frac{E}{(t+1)^3}$$

za vježbu:

a) $\int\frac{1-\sqrt{x^2+x+1}}{x\cdot\sqrt{x^2+x+1}}\,dx$ (racionalisati

b) $\int x\cdot\sqrt{x^2-2x+2}\,dx$

d) $\int\frac{x^2\,dx}{(4-2x+x^2)\cdot\sqrt{2+2x-x^2}}$

c) $\int\frac{dx}{\left[1+\sqrt{x\cdot(1+x)}\right]^2}$

9. Integracija nekih transcedentnih (nealgebarskih) f-ja

U ovoj lekciji ćemo posmatrati sljedeće tipove integrala (R predstavlja racionalnu f-ju)

I. $\int R(\sin x, \cos x)\,dx$ – uvodimo smjenu $z = tg\frac{x}{2}$,

pri čemu je $\sin x = \frac{2z}{1+z^2}$, $\cos x = \frac{1-z^2}{1+z^2}$, $dx = \frac{2dz}{1+z^2}$

II. $\int R(tg x)\,dx$ – uvodimo smjenu $tg x = z$,

pri čemu je $x = arc\,tg\,z$, $dx = \frac{dz}{1+z^2}$

III. $\int R(e^x)\,dx$ – uvodimo smjenu $e^x = z$,

pri čemu je $x = \ln z$, $dx = \frac{dz}{z}$.

$$z = tg\frac{x}{2} \Rightarrow \frac{x}{2} = arc\,tg\,z \Rightarrow x = 2\,arc\,tg\,z$$

$$dx = \frac{2\,dz}{1+z^2}, \quad \sin x = \sin 2\cdot\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \begin{matrix} /:\cos^2\frac{x}{2} \\ /:\cos^2\frac{x}{2}\end{matrix} =$$

$$= \frac{2\,tg\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{2z}{1+z^2} \quad tj. \quad \sin x = \frac{2z}{1+z^2}$$

$$\cos x = \cos 2\cdot\frac{x}{2} = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \begin{matrix} /:\cos^2\frac{x}{2} \\ /:\cos^2\frac{x}{2}\end{matrix} = \frac{1-z^2}{1+z^2}$$

(#) Odrediti integrale

a) $\int \frac{dx}{2\sin x - \cos x}$; b) $\int \frac{dx}{5+4\cos ax}$;

c) $\int \frac{tg x\,dx}{1-ctg^2 x}$; d) $\int \frac{e^{3x}\,dx}{e^{2x}+1}$.

Rj.

a)
$$\int \frac{dx}{2\sin x - \cos x} = \left| \begin{matrix} tg\frac{x}{2} = z & \sin x = \frac{2z}{1+z^2} \\ dx = \frac{2dz}{1+z^2} & \cos x = \frac{1-z^2}{1+z^2} \end{matrix} \right| =$$

$$= \int \frac{\frac{2dz}{1+z^2}}{\frac{4z}{1+z^2} - \frac{1-z^2}{1+z^2}} = \int \frac{2\,dz}{z^2+4z-1} \overset{(*)}{=} 2\int \frac{d(z+2)}{(z+2)^2-5} =$$

$$z^2+4z-1 = z^2+2\cdot z\cdot 2+4-4-1 = (z+2)^2-5$$
$$d(z+2) = dz \qquad \ldots(*)$$

$$= 2\cdot\frac{1}{2\sqrt{5}}\ln\left|\frac{z+2-\sqrt{5}}{z+2+\sqrt{5}}\right| + C = \frac{1}{\sqrt{5}}\ln\left|\frac{tg\frac{x}{2}+2-\sqrt{5}}{tg\frac{x}{2}+2+\sqrt{5}}\right| + C$$

b)
$$\int \frac{dx}{5+4\cos ax} = \left| \begin{matrix} tg\frac{ax}{2} = z \Rightarrow \frac{ax}{2} = arc\,tg\,z \\ dx = \frac{2\,dz}{a(1+z^2)}, \quad \cos ax = \frac{1-z^2}{1+z^2} \end{matrix} \right| =$$

$$= \int \frac{\frac{2\,dz}{a(1+z^2)}}{5+\frac{4(1-z^2)}{1+z^2}} = \frac{2}{a}\int \frac{dz}{5+5z^2+4-4z^2} = \frac{2}{a}\int \frac{dz}{z^2+9} =$$

$$= \frac{2}{a} \cdot \frac{1}{3} \operatorname{arctg} \frac{z}{3} + c = \frac{2}{3a} \operatorname{ar\ tg}\left(\frac{1}{3} \operatorname{tg} \frac{ax}{2}\right) + c$$

c) $$\int \frac{\operatorname{tg} x \, dx}{1 - \operatorname{ctg}^2 x} = \left| \begin{array}{ll} \operatorname{tg} x = z, & x = \operatorname{arctg} z \\ dx = \frac{dz}{1+z^2} & \operatorname{ctg}^3 x = \left(\frac{1}{\operatorname{tg} x}\right)^2 = \frac{1}{z^2} \end{array} \right|$$

$$= \int \frac{\frac{z\,dz}{1+z^2}}{1 - \frac{1}{z^2}} = \int \frac{\frac{z\,dz}{1+z^2}}{\frac{z^2-1}{z^2}} = \int \frac{z^3\,dz}{z^4-1} =$$

$$= \int \frac{\frac{1}{4}\, d(z^4-1)}{z^4-1} = \frac{1}{4} \ln|z^4-1| + c = \frac{1}{4} \ln|\operatorname{tg}^4 x - 1| + c$$

d) $$\int \frac{e^{3x}\,dx}{e^{2x}+1} = \left| \begin{array}{c} e^x = z \\ e^x dx = dz \\ dx = \frac{dz}{z} \end{array} \right| = \int \frac{z^3 \cdot \frac{dz}{z}}{z^2+1} =$$

$$= \int \frac{z^2\,dz^{+1-1}}{z^2+1} = \int \left(1 - \frac{1}{z^2+1}\right) dz = \int dz - \int \frac{dz}{z^2+1}$$

$$= z - \operatorname{arctg} z + c = e^x - \operatorname{arctg} e^x + c$$

Zadaci za vježbu

① $\displaystyle\int \frac{\cos x \, dx}{1+\cos x}$

② $\displaystyle\int \frac{dx}{\sin kx}$

③ $\displaystyle\int \frac{dx}{\sin^3 x}$

④ $\displaystyle\int \frac{dx}{4\cos x + 3\sin x}$

⑤ $\displaystyle\int \operatorname{tg}^5 3x \, dx$

⑥ $\displaystyle\int \frac{dx}{1+\operatorname{tg} x}$

⑦ $\displaystyle\int \frac{e^{2t} - 2e^t}{1+e^{2t}} dt$

⑧ $\displaystyle\int \frac{e^x - 1}{e^x + 1} dx$

⑨* $\displaystyle\int \frac{1+\operatorname{tg} x}{\sin 2x} dx$

⑩* $\displaystyle\int \frac{e^{2x}\,dx}{(2+e^x+e^{-x})^2}$

Rješenja

1. $x - \operatorname{tg}\frac{x}{2}$

2. $\frac{1}{k} \left|\operatorname{tg}\frac{kx}{2}\right|$

3. $\frac{1}{2}\left(\ln\left|\operatorname{tg}\frac{x}{2}\right| - \frac{\cos x}{\sin^2 x}\right)$

4. $\frac{1}{5} \ln\left|\frac{1+2\operatorname{tg}\frac{x}{2}}{2-\operatorname{tg}\frac{x}{2}}\right|$

5. $\frac{1}{12}\operatorname{tg}^4 3x - \frac{1}{6}\operatorname{tg}^2 3x - \frac{1}{3}\ln|\cos 3x|$

6. $\frac{1}{2}(x + \ln|\sin x + \cos x|)$

7. $\frac{1}{2} \ln(e^{2t}+1) - 2\operatorname{arctg} e^t$

8. $2\ln(e^x+1) - x$

9. $\frac{1}{2}(\operatorname{tg} x + \ln|\operatorname{tg} x|)$

10. $\ln(e^x+1) + \frac{18e^{2x}+27e^x+11}{6(e^x+1)^3}$

Izabrani **Zadaci za vježbu** sa rješenjima
(iz lekcije Integracija nekih nealgebarskih funkcija)

$\int R(\sin x, \cos x)\,dx$, R - racionalna f-ja

koristimo smjenu $tg\frac{x}{2} = t \Rightarrow$

$$\Rightarrow dx = \frac{2dt}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$tg\frac{x}{2} = t$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2} \;\; :\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2} \;\; :\cos^2\frac{x}{2}} = \frac{2tg\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{2t}{t^2+1}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2} \;\; :\cos^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2} \;\; :\cos^2\frac{x}{2}} = \frac{1-tg^2\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{1-t^2}{1+t^2}$$

$$tg\frac{x}{2} = t \Rightarrow \frac{x}{2} = arctg\, t \Rightarrow x = 2arctg\, t \Rightarrow dx = \frac{2dt}{1+t^2}$$

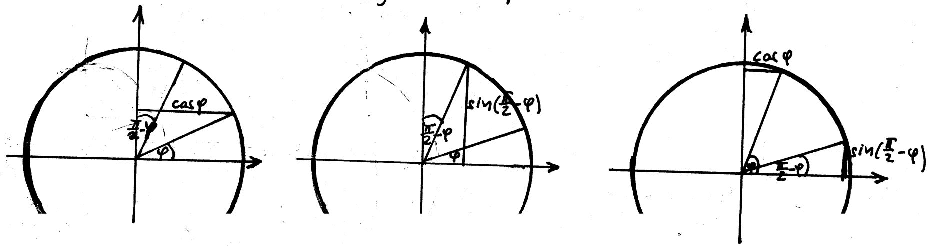
①
$$\int\frac{dx}{\sin x} = \left|\begin{array}{l} tg\frac{x}{2} = t \\ dx = \frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2}\end{array}\right| = \int\frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int\frac{dt}{t} = \ln|t| + c =$$
$$= \ln\left|tg\frac{x}{2}\right| + c$$

②
$$\int\frac{dx}{\cos x} = \left|\begin{array}{c} \sin x = \cos(\frac{\pi}{2}-x) \\ \cos x = \sin(\frac{\pi}{2}-x) \\ \text{OBJASNI} \\ \text{OVO}\end{array}\right| = \int\frac{dx}{\sin(\frac{\pi}{2}-x)} = \left|\begin{array}{l} \frac{\pi}{2}-x = t \\ -dx = dt \\ dx = -dt\end{array}\right| =$$

$$= -\int\frac{dt}{\sin t} \overset{1.zad.}{=\!=} -\ln\left|tg\frac{t}{2}\right| + c = -\ln\left|tg(\frac{\pi}{4}-\frac{x}{2})\right| + C =$$

$$= \ln\left|tg(\frac{\pi}{4}-\frac{x}{2})\right|^{-1} = \ln\left|ctg(\frac{\pi}{4}-\frac{x}{2})\right| + C = \left|\begin{array}{l} ctg\, x = \frac{\cos x}{\sin x} = \\ = \frac{\sin(\frac{\pi}{2}-x)}{\cos(\frac{\pi}{2}-x)} = tg(\frac{\pi}{2}-x)\end{array}\right|$$

$$= \ln\left|tg\,\frac{\pi}{2} - (\frac{\pi}{4}-\frac{x}{2})\right| + c = \ln\left|tg(\frac{x}{2}+\frac{\pi}{4})\right| + c$$



③
$$\int\frac{dx}{5-4\sin x+3\cos x} = \left|\begin{array}{l} tg\frac{x}{2} = t \\ dx = \frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2}\end{array}\right| = \int\frac{\frac{2dt}{1+t^2}}{5-4\cdot\frac{2t}{1+t^2}+3\cdot\frac{1-t^2}{1+t^2}} =$$

$$= 2\int\frac{\frac{dt}{\cancel{1+t^2}}}{\frac{5+5t^2-8t+3-3t^2}{\cancel{1+t^2}}} = 2\int\frac{dt}{2t^2-8t+8} = \int\frac{dt}{t^2-4t+4}$$

$$= \int\frac{dt}{(t-2)^2} = \left|\begin{array}{l} t-2 = z \\ dt = dz\end{array}\right| = \int\frac{dz}{z^2} = -\frac{1}{z} + C = -\frac{1}{t-2} + C = -\frac{1}{tg\frac{x}{2}-2} + C$$

④ $^{\vee}$
$$I = \int\frac{\cos x+2\sin x}{4\cos x+3\sin x}dx = \left|\begin{array}{ll} tg\frac{x}{2} = t & \cos x = \frac{1-t^2}{1+t^2} \\ dx = \frac{2dt}{1+t^2} & \sin x = \frac{2t}{1+t^2}\end{array}\right| =$$

$$= \int\frac{\frac{1-t^2}{1+t^2}+2\frac{2t}{1+t^2}}{4\cdot\frac{1-t^2}{1+t^2}+3\frac{2t}{1+t^2}}\cdot\frac{2dt}{1+t^2} = \int\frac{\frac{1-t^2+4t}{1+t^2}}{\frac{4-4t^2+6t}{1+t^2}}\cdot\frac{2dt}{1+t^2} =$$

$$= \int\frac{-t^2+4t+1}{(-2t^2+3t+2)(1+t^2}dt = \ldots$$

⑤ $^{\vee}$
$$\int\frac{dx}{8-4\sin x+7\cos x}$$

⑥ $^{\vee}$
$$\int\frac{\cos x+\sin x}{\cos x-2\sin x}dx$$

$$\int R(\sin^2x,\ \sin x\cos x,\ \cos^2x)\,dx$$

R- racionalna f-ja

ili $\int R(\text{tg}\,x)\,dx$

uvodimo smjenu $\text{tg}\,x = t \Rightarrow$

$$\Rightarrow dx = \frac{dt}{1+t^2},\quad \sin^2x = \frac{t^2}{1+t^2},\quad \cos^2x = \frac{1}{1+t^2},$$

$$\sin x\cos x = \frac{t}{1+t^2}$$

$$\text{tg}\,x = t \Rightarrow x = \text{arc tg}\,t \Rightarrow dx = \frac{dt}{1+t^2}$$

$$\sin^2x = \frac{\sin^2x}{\sin^2x+\cos^2x}\begin{matrix}:\cos^2x\\ :\cos^2x\end{matrix} = \frac{\text{tg}^2x}{\text{tg}^2x+1} = \frac{t^2}{1+t^2}$$

$$\cos^2x = \frac{\cos^2x}{\sin^2x+\cos^2x}\begin{matrix}:\cos^2x\\ :\cos^2x\end{matrix} = \frac{1}{\text{tg}^2x+1} = \frac{1}{1+t^2}$$

$$\sin x\cos x = \frac{\sin x\cos x}{\sin^2x+\cos^2x}\begin{matrix}:\cos^2x\\ :\cos^2x\end{matrix} = \frac{\text{tg}\,x}{\text{tg}^2x+1} = \frac{t}{1+t^2}$$

① $$\int\frac{dx}{\cos^4x} = \left|\begin{matrix}\text{tg}\,x = t & \cos^2x = \frac{1}{1+t^2}\\ dx = \frac{dt}{1+t^2} & \end{matrix}\right| = \int\frac{\frac{dt}{1+t^2}}{\left(\frac{1}{1+t^2}\right)^2} = \int\frac{(1+t^2)^{\cancel{2}}}{\cancel{1+t^2}}dt =$$

$$= \int(1+t^2)dt = \int dt + \int t^2dt = t + \frac{t^3}{3} + C = \text{tg}\,x + \frac{1}{3}\text{tg}^3x + C$$

② $$\int\frac{dx}{\sin^2x - 4\sin x\cos x + 5\cos^2x} = \left|\begin{matrix}\text{tg}\,x = t & dx = \frac{dt}{1+t^2} & \sin x\cdot\cos x =\\ \sin^2x = \frac{t^2}{1+t^2} & \cos^2x = \frac{1}{1+t^2} & = \frac{t}{1+t^2}\end{matrix}\right|$$

$$= \int\frac{\frac{dt}{1+t^2}}{\frac{t^2}{1+t^2} - 4\cdot\frac{t}{1+t^2} + 5\cdot\frac{1}{1+t^2}} = \int\frac{dt}{t^2-4t+5} = \int\frac{dt}{(t-2)^2+1} = \text{arctg}(t-2)+C$$

$$= \text{arc tg}(\text{tg}\,x - 2) + C$$

③ $$\int\text{tg}^3x\,dx = \left|\begin{matrix}\text{tg}\,x = t\\ dx = \frac{dt}{1+t^2}\end{matrix}\right| = \int t^3\cdot\frac{dt}{1+t^2} = \int\frac{t^3\ +t-t}{1+t^2}dt = \int\frac{t+t^3}{1+t^2}dt -$$

$$-\int\frac{t}{1+t^2}dt = \int\frac{\cancel{t(1+t^2)}}{\cancel{(1+t^2)}}dt - \frac{1}{2}\int\frac{2t\,dt}{1+t^2} = \left|\begin{matrix}t^2 = s\\ 2t\,dt = ds\end{matrix}\right| =$$

$$= \int t\,dt - \frac{1}{2}\int\frac{ds}{1+s} = \frac{t^2}{2} - \frac{1}{2}\ln|1+s| + C = \frac{1}{2}t^2 - \frac{1}{2}\ln|t^2+1| + C =$$

$$= \frac{1}{2}\text{tg}^2x - \frac{1}{2}\ln|\text{tg}^2+1| + C = \frac{1}{2}\text{tg}^2x - \frac{1}{2}\ln\left|\frac{\sin^2x}{\cos^2x}+1\right| + C = \frac{1}{2}\text{tg}^2x - \frac{1}{2}\ln\left|\frac{1}{\cos^2x}\right| + C$$

$$= \frac{1}{2}\text{tg}^2x + \ln\left|\frac{1}{\cos^2x}\right|^{-\frac{1}{2}} + C = \frac{1}{2}\text{tg}^2x + \ln|\cos^2x|^{\frac{1}{2}} + C = \frac{1}{2}\text{tg}^2x + \ln|\cos x| + C$$

④$^{\vee}$ $$\int\frac{\cos x + 2\sin x}{4\cos x + 3\sin x}\begin{matrix}:\cos x\\ \\ :\cos x\end{matrix}\,dx = \int\frac{1+2\,\text{tg}\,x}{4+3\,\text{tg}\,x}dx = \left|\begin{matrix}\text{tg}\,x = t\\ dx = \frac{dt}{1+t^2}\end{matrix}\right| = \int\frac{1+2t}{4+3t}\cdot\frac{dt}{1+t^2}$$

$$\frac{1+2t}{(4+3t)(1+t^2)} = \frac{a}{4+3t} + \frac{bt+c}{1+t^2}$$

...

⑤$^{\vee}$ $$\int\frac{dx}{\sin^4x}$$

⑥$^{\vee}$ $$\int\frac{dx}{3\cos^2x + 4\sin^2x}$$

⑦$^{\vee}$ $$\int\frac{\text{tg}\,x}{\text{tg}^2x - 2\,\text{tg}\,x - 3}dx \to \text{Rj.}\ -\frac{1}{10}x + \frac{3}{40}\ln|\text{tg}\,x-3| + \frac{1}{8}\ln|\text{tg}\,x+1| + \frac{1}{5}\ln|\cos x| + C$$

$\#$ Izračunati integral $I=\int \frac{dx}{3\cos^2 x+4\sin^2 x}$.

Rj. $tg\, x = t$

$x = arc\, tg\, t$

$dx = \frac{dt}{1+t^2}$

$$\sin^2 x = \frac{\sin^2 x}{\sin^2 x+\cos^2 x} \begin{matrix} :\cos^2 x \\ :\cos^2 x \end{matrix} = \frac{tg^2 x}{tg^2 x+1} = \frac{t^2}{1+t^2}$$

$$\cos^2 x = \frac{\cos^2 x}{\sin^2 x+\cos^2 x} \begin{matrix} :\cos^2 x \\ :\cos^2 x \end{matrix} = \frac{1}{tg^2 x+1} = \frac{1}{1+t^2}$$

$$I=\int \frac{dx}{3\cos^2 x+4\sin^2 x} = \left| \begin{matrix} tg\,x=t & \sin^2 x = \frac{t^2}{1+t^2} \\ dx=\frac{dt}{1+t^2} & \cos^2 x = \frac{1}{1+t^2} \end{matrix} \right| =$$

$$= \int \frac{\frac{dt}{1+t^2}}{\frac{3}{1+t^2}+\frac{4t^2}{1+t^2}} = \int \frac{\frac{dt}{1+t^2}}{\frac{3+4t^2}{1+t^2}} = \int \frac{dt}{3+4t^2} = \int \frac{dt}{(\sqrt{3})^2+(2t)^2}$$

$$= \left| \begin{matrix} 2t=\sqrt{3}u \\ 2dt=\sqrt{3}du \\ dt=\frac{\sqrt{3}}{2}du \\ u=\frac{2t}{\sqrt{3}} \end{matrix} \right| = \int \frac{\frac{\sqrt{3}}{2}du}{3+3u^2} = \frac{\sqrt{3}}{2}\cdot\frac{1}{3}\int\frac{du}{1+u^2} = \frac{\sqrt{3}}{6}\, arc\,tg\, u + c =$$

$$= \frac{\sqrt{3}}{6}\, arc\,tg\,\frac{2t}{\sqrt{3}} + c = \frac{\sqrt{3}}{6}\, arc\,tg\,\frac{2tgx}{\sqrt{3}} + c$$

$\#$ Odrediti $I=\int x^2 \sin x\, dx$.

Rj.

$$I=\int x^2 \sin x\, dx = \left| \begin{matrix} u=x^2 & dv=\sin x\, dx \\ du=2x\,dx & v=-\cos x \end{matrix} \right| =$$

$$= -x^2\cos x - \int(-\cos x)\cdot 2x\,dx = -x^2+2\int x\cos x\, dx$$

$$\int x\cos x\, dx = \left| \begin{matrix} u=x & dv=\cos x\,dx \\ du=dx & v=\sin x \end{matrix} \right| = x\sin x - \int \sin x\, dx$$

$$= x\sin x - (-\cos x) + C = x\sin x + \cos x + C_1$$

$$I = -x^2\cos x + 2(x\sin x+\cos x + C_1) = -x^2\cos x + 2x\sin x + 2\cos x + C$$

$$= 2x\sin x - (x^2-2)\cos x + C$$

$\#$ Izračunati integral $\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}dx$.

Rj. Uvodimo smjenu $tg\frac{x}{2}=t \Rightarrow \frac{x}{2}=arc\,tg\,t$

$\sin 2x = 2\sin x\cos x$

$x = 2\,arc\,tg\,t$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} =$

$dx = \frac{2}{1+t^2}$

$$= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{1} = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \begin{matrix} /:\cos^2\frac{x}{2} \\ /:\cos^2\frac{x}{2} \end{matrix} = \frac{2tg\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{2t}{t^2+1}$$

$\cos 2x = \cos^2 x - \sin^2 x$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{1} = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}} \begin{matrix} /:\cos^2\frac{x}{2} \\ /:\cos^2\frac{x}{2} \end{matrix} =$$

$$= \frac{1-tg^2\frac{x}{2}}{tg^2\frac{x}{2}+1} = \frac{1-t^2}{1+t^2}$$

$$\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}\,dx = \left| \begin{matrix} tg\frac{x}{2}=t & \sin x=\frac{2t}{1+t^2} \\ dx=\frac{2}{1+t^2} & \cos x=\frac{1-t^2}{1+t^2} \\ x=2\,arc\,tg\,t & \end{matrix} \right| =$$

$$= \int \frac{1-\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}{1+\frac{2t}{1+t^2}-\frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt = \int \frac{\frac{\underline{\underline{1}}+\underline{\underline{t^2}}-2t+1\underline{\underline{-t^2}}}{\cancel{1+t^2}}}{\frac{\underline{\underline{1}}+t^2+2t\,\underline{\underline{-1}}+t^2}{\cancel{1+t^2}}} \cdot \frac{2}{1+t^2}\,dt$$

$$=2\int \frac{2-2t}{2t^2+2t} \cdot \frac{1}{1+t^2}\,dt = 2\int \frac{1-t}{(t^2+t)(1+t^2)}\,dt = 2\int \frac{1-t}{t(t+1)(t^2+1)}\,dt$$

$$\frac{1-t}{t(t+1)(t^2+1)} = \frac{A}{t} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1} \quad / \cdot t(t+1)(t^2+1)$$

$$-t+1 = A\underbrace{(t+1)(t^2+1)}_{t^3+t^2+t+1} + B\underbrace{(t^2+1)\cdot t}_{t^3+t} + (Ct+D)\underbrace{t(t+1)}_{t^2+t}$$

$$-t+1 = A(t^3+t^2+t+1) + B(t^3+t) + C(t^3+t^2) + D(t^2+t)$$

$$\begin{array}{l} A+B+C \qquad = 0 \\ A+C+D \qquad = 0 \\ A+B \qquad +D = -1 \\ \hline A \qquad\qquad = 1 \end{array} \qquad \begin{array}{ll} B+C \qquad = -1 & (a) \\ \quad C+D = -1 & (b) \\ B \qquad +D = -2 & (c) \\ \hline \end{array} \qquad \begin{array}{l} (a):\ B+C = -1 \\ (c)-(b):\ B-C = -1 \quad + \\ \hline 2B = -2 \\ B = -1 \end{array}$$

$$-1+D=-2$$
$$D=-1 \qquad C-1=-1$$
$$C=0$$

$$A=1 \qquad C=0$$
$$B=-1 \qquad D=-1$$

$$2\int \frac{1-t}{t(t+1)(t^2+1)}\,dt = 2\int \left(\frac{1}{t} + \frac{(-1)}{t+1} + \frac{(-1)}{t^2+1}\right) dt =$$

$$= 2\ln|t| - 2\ln|t+1| - 2\,arc\,tg\,t + c =$$

$$= 2\ln|tg\tfrac{x}{2}| - 2\ln|tg\tfrac{x}{2}+1| - 2\,arc\,tg\,|tg\tfrac{x}{2}| + c$$

Dio tablice integrala

1. $\int u^a\,du = \frac{u^{a+1}}{a+1}+C,\ a \neq -1.$

2. $\int u^{-1}\,du = \int \frac{du}{u} = \int \frac{u'}{u}\,dx = \ln|u|+C.$

3. $\int a^u\,du = \frac{a^u}{\ln a}+C;\ \int e^u\,du = e^u+C.$

4. $\int \sin u\,du = -\cos u+C.$

5. $\int \cos u\,du = \sin u+C.$

6. $\int \sec^2 u\,du = \operatorname{tg} u+C.$

7. $\int \operatorname{cosec}^2 u\,du = -\operatorname{ctg} u+C.$

8. $\int \frac{du}{u^2+a^2} = \frac{1}{a}\operatorname{arc\,tg}\frac{u}{a}+C.$

9. $\int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right|+C.$

10. $\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a}+C.$

11. $\int \frac{du}{\sqrt{u^2+a}} = \ln\left|u+\sqrt{u^2+a}\right|+C.$

Sveska je skinuta sa stranice *pf.unze.ba\nabokov*
U svesci je moguća pojava grešaka - za uočene greške pisati na **infoarrt@gmail.com**

Određeni integrali

Osobine određenih integrala su:

1∘ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2∘ $\int_a^a f(x)\,dx = 0$

3∘ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

4∘ $\int_a^b [f_1(x) + f_2(x) - f_3(x)]\,dx = \int_a^b f_1(x)\,dx + \int_a^b f_2(x)\,dx - \int_a^b f_3(x)\,dx$

5∘ $\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx$, gdje je α konstanta

Određene integrale ćemo računati pomoću Njutn-Lejbicove formule

$$\int_a^b f(x)\,dx = \int f(x)\,dx \Big|_a^b = F(x)\Big|_a^b = F(b) - F(a)$$

gdje je $F'(x) = f(x)$

(#) Izračunati integrale

a) $\int_2^3 3x^2 dx$; b) $\int_0^4 (1+e^{\frac{x}{4}})\,dx$; c) $\int_{-1}^7 \frac{dt}{\sqrt{3t+4}}$;

d) $\int_0^{\pi/2a} (x+3)\sin ax$.

Rj:

a) $$\int_2^3 3x^2 dx = 3\int_2^3 x^2 dx = 3\cdot\frac{x^3}{3}\Big|_2^3 = x^3\Big|_2^3 = 3^3 - 2^3 = 27-8 = 19$$

b) $$\int_0^4 (1+e^{\frac{x}{4}})\,dx = \int_0^4 dx + \int_0^4 e^{\frac{x}{4}} dx = \int_0^4 dx + 4\int_0^4 e^{\frac{x}{4}} d\left(\frac{x}{4}\right) =$$

$$= x\Big|_0^4 + 4\,e^{\frac{x}{4}}\Big|_0^4 = (4-0) + 4\left(e^{\frac{4}{4}} - e^{\frac{0}{4}}\right) = 4 + 4e - 4 = 4e$$

c) $$\int_{-1}^7 \frac{dt}{\sqrt{3t+4}} = \int_{-1}^7 (3t+4)^{-\frac{1}{2}} dt = \begin{vmatrix} d(3t+4) = 3\,dt \\ dt = \frac{1}{3} d(3t+4) \end{vmatrix} =$$

$$= \frac{1}{3}\int_{-1}^7 (3t+4)^{-\frac{1}{2}} d(3t+4) = \frac{2}{3}(3t+4)^{\frac{1}{2}}\Big|_{-1}^7 = \frac{2}{3}\left(\sqrt{25} - \sqrt{1}\right) = \frac{8}{3}$$

d) $$\int_0^{\frac{\pi}{2a}} (x+3)\sin ax\,dx = \begin{vmatrix} u = x+3 & dv = \sin ax\,dx \\ du = dx & v = \frac{1}{a}\int \sin ax\,d(ax) = \frac{-1}{a}\cos ax \end{vmatrix} =$$

$$= -\frac{1}{a}(x+3)\cos ax\Big|_0^{\frac{\pi}{2a}} + \frac{1}{a}\int_0^{\pi/2a} \cos ax\,dx = -\frac{1}{a}\left[\left(\frac{\pi}{2a}+3\right)\underbrace{\cos\frac{\pi}{2}}_{=0} - 3\underbrace{\cos 0}_{=1}\right] +$$

$$+ \frac{1}{a}\cdot\frac{1}{a}\int_0^{\pi/2a} \cos ax\,d(ax) = \frac{3}{a} + \frac{1}{a^2}\underbrace{\sin ax\Big|_0^{\frac{\pi}{2a}}}_{\sin\frac{\pi}{2} - \sin 0} = \frac{3}{a} + \frac{1}{a^2} = \frac{1+3a}{a^2}$$

Zadaci za vežbu

① $\displaystyle\int_1^5 \frac{dx}{3x-2}$

② $\displaystyle\int_0^1 \frac{dz}{(2z+1)^3}$

③ $\displaystyle\int_1^2 \frac{dt}{t^2+5t+4}$

④ $\displaystyle\int_0^2 \frac{x+3}{x^2+4}\,dx$

⑤ $\displaystyle\int_{-a}^{a} x\cos\frac{x}{a}\,dx$

⑥ $\displaystyle\int_0^{\pi} \cos\frac{x}{2}\cos\frac{3x}{2}\,dx$

⑦* $\displaystyle\int_{-\pi}^{\pi} x\sin x\cos x\,dx$

⑧ $\displaystyle\int_1^{e} (1+\ln y)^2\,dy$

Rešenja:

1. $\frac{\ln 13}{3}$ 2. $\frac{2}{9}$ 3. $\frac{1}{3}\ln\frac{5}{4}$ 4. $\frac{3\pi}{8}+\frac{\ln 2}{2}$

5. 0 6. 0 7. $-\frac{\pi}{2}$ 8. $2e-1$

Zamena promenljivih u određenom integralu

$$\int_a^b f(x)\,dx = \left|\begin{array}{llll} x=\varphi(t) & x=a \Rightarrow & a=\varphi(\alpha) \Rightarrow & t=\alpha \\ dx=\varphi'(t)\,dt & x=b \Rightarrow & b=\varphi(\beta) \Rightarrow & t=\beta \end{array}\right|$$

$$= \int_\alpha^\beta f(\varphi(t))\,\varphi'(t)\,dt = \int_\alpha^\beta F(t)\,dt$$

(#) Izračunati integrale

a) $\displaystyle\int_0^5 \frac{x\,dx}{\sqrt{1+3x}}$; b) $\displaystyle\int_{\ln 2}^{\ln 3} \frac{dx}{e^x-e^{-x}}$; c) $\displaystyle\int_1^{\sqrt{3}} \frac{(x^3+1)\,dx}{x^2\sqrt{4-x^2}}$; d) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$.

Rj.

a)
$$\int_0^5 \frac{x\,dx}{\sqrt{1+3x}} = \left|\begin{array}{ll} 1+3x=t^2 & 3\,dx=2t\,dt \\ \sqrt{1+3x}=t & dx=\frac{2}{3}t\,dt \\ 3x=t^2-1 & \\ x=\frac{t^2-1}{3} & x\Big|_0^5 \Rightarrow t\Big|_1^4 \end{array}\right| = \int_1^4 \frac{\frac{t^2-1}{3}\cdot\frac{2}{3}t\,dt}{t} =$$

$$= \frac{2}{9}\int_1^4 (t^2-1)\,dt = \frac{2}{9}\left(\frac{t^3}{3}\Big|_1^4 - t\Big|_1^4\right) = \frac{2}{9}\left(\frac{1}{3}(64-1)-(4-1)\right) =$$

$$= \frac{2}{9}\left(\frac{63}{3}-3\right) = \frac{2}{9}(21-3) = 4$$

b)
$$\int_{\ln 2}^{\ln 3} \frac{dx}{e^x-e^{-x}} = \left|\begin{array}{ll} e^x=t & e^{-x}=t^{-1}=\frac{1}{t} \\ x=\ln t & \\ dx=\frac{dt}{t} & x\Big|_{\ln 2}^{\ln 3} \Rightarrow t\Big|_2^3 \end{array}\right| = \int_2^3 \frac{\frac{dt}{t}}{t-\frac{1}{t}} = \int_2^3 \frac{\frac{dt}{t}}{\frac{t^2-1}{t}} =$$

$$= \int_2^3 \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\Bigg|_2^3 = \frac{1}{2}\left(\ln\frac{2}{4}-\ln\frac{1}{3}\right) = \frac{1}{2}\cdot\ln\frac{\frac{1}{2}}{\frac{1}{3}} = \frac{\ln\frac{3}{2}}{2}$$

c) $$\int_{1}^{\sqrt{3}} \frac{(x^3+1)\,dx}{x^2\sqrt{4-x^2}} = \left| \begin{array}{l} x = 2\sin t \\ dx = 2\cos t\,dt \quad x\Big|_{1}^{\sqrt{3}} \Rightarrow t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ x^3 = 8\sin^3 t \\ \sqrt{4-x^2} = \sqrt{4-4\sin^2 t} = \sqrt{4(1-\sin^2 t)} \end{array} \right| =$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(8\sin^3 t+1)\,2\cos t\,dt}{4\sin^2 t\sqrt{4\cos^2 t}} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\sin^3 t+1}{4\sin^2 t}\,dt =$$

$$= 2\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin t\,dt + \frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dt}{\sin^2 t} = -2\cos t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} - \frac{1}{4}\operatorname{ctg} t\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}} =$$

$$= -2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}\right) - \frac{1}{4}\left(\frac{\sqrt{3}}{3} - \sqrt{3}\right) = \frac{7}{2\sqrt{3}} - 1$$

d) $$\int_{0}^{\frac{\pi}{2}} \frac{dx}{2+\cos x} = \left| \begin{array}{ll} \operatorname{tg}\frac{x}{2} = z & dx = \frac{2dz}{1+z^2} \\ \cos x = \frac{1-z^2}{1+z^2} & x\Big|_{0}^{\frac{\pi}{2}} \Rightarrow z\Big|_{0}^{1} \end{array} \right| =$$

$$= \int_{0}^{1} \frac{\frac{2dz}{1+z^2}}{2+\frac{1-z^2}{1+z^2}} = 2\int_{0}^{1} \frac{dz}{z^2+3} = \frac{2}{\sqrt{3}}\operatorname{arctg}\frac{z}{\sqrt{3}}\Big|_{0}^{1} =$$

$$= \frac{2}{\sqrt{3}} \cdot \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

(#) Dokazati da za parnu f-ju f(x) vrijedi

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

dok za neparnu f-ju f(x) vrijedi $\int_{-a}^{a} f(x)\,dx = 0$.

Rj. Prvo rastavimo interval [-a, a] na dva dijela [-a,0] i [0,a]

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx \qquad \ldots(*)$$

Posmatrajmo sad $\int_{-a}^{0} f(x)\,dx$. Ako uvedemo smjenu $x=-z$ imamo da je $dx=-dz$ i $z_1=a$ za $x_1=-a$, $z_2=0$ za $x_2=0$

$$\int_{-a}^{0} f(x)\,dx = -\int_{a}^{0} f(-z)\,dz = \int_{0}^{a} f(-z)\,dz = \left|\overset{\text{nove prom}}{x=z}\right| = \int_{0}^{a} f(-x)\,dx$$

Prema tome (*) sad postaje

$$I = \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx = \int_{0}^{a} [f(x)+f(-x)]\,dx$$

Za parnu f-ju znamo da $f(-x)=f(x)$ dok je za neparnu f-ju $f(-x)=-f(x)$. Prema tome

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_{0}^{a} f(x)\,dx, & \text{ako je } f(x) \text{ parna f-ja} \\ 0, & \text{ako je } f(x) \text{ neparna f-ja} \end{cases}$$

Znamo da za parnu f-ju f(x) vrijedi

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx,$$

dok za neparnu f-ju f(x) vrijedi: $\int_{-a}^{a} f(x)\,dx = 0.$

Iskoristiti ovu osobinu i izračunati sljedeće integrale:

a) $\int_{-\sqrt{5}}^{\sqrt{5}} (3x - 2x^5)\,dx$ b) $\int_{-\pi}^{\pi} \sin^7 2x\,dx$ c) $\int_{3}^{-3} t^8 \arcsin t\,dt$

d) $\int_{-2}^{2} \frac{x^5 + 7x^4 + x^3 - 5x^2 - 2}{x^3 + x}\,dx$

Rj.

a) $f(x) = 3x - 2x^5$

$f(-x) = 3(-x) - 2(-x)^5 = -3x + 2x^5 = -(3x - 2x^5) = -f(x)$

$$\int_{-\sqrt{5}}^{\sqrt{5}} (3x - 2x^5)\,dx = \left| \begin{array}{l} \text{primijetimo} \\ \text{da je} \\ 3x-2x^5 \text{ neparna} \\ \text{f-ja} \end{array} \right| = 0$$

b) $f(x) = \sin^7 2x \Rightarrow f(-x) = (\sin 2(-x))^7 = (-\sin 2x)^7 = -\sin^7 2x = -f(x)$

Kako je $\sin^7 2x$ neparna f-ja $\int_{-\pi}^{\pi} \sin^7 2x\,dx = 0$

c) $\int_{3}^{-3} t^8 \arcsin t\,dt = 0$ ZAŠTO? OBJASNITI!

d) $$\int_{-2}^{2} \frac{x^5 + 7x^4 + x^3 - 5x^2 - 2}{x^3 + x}\,dx = \int_{-2}^{2} \frac{\overbrace{x^5 + x^3}^{x^2(x^3+x)}}{x^3 + x}\,dx + \int_{-2}^{2} \frac{7x^4 - 5x^2 - 2}{x^3 - x}\,dx =$$

$$= \int_{-2}^{2} x^2\,dx + 0 = 2\int_{0}^{2} x^2\,dx = 2\,\frac{x^3}{3}\Big|_{0}^{2} = \frac{16}{3}$$

Zadaci za vježbu

Izračunati integrale

① $\int_{0}^{1} \frac{x^2\,dx}{(x+1)^4}$ uvođenjem smjene $x + 1 = z$.

② $\int_{0}^{\ln 2} \sqrt{e^x - 1}\,dx$ uvođenjem smjene $\sqrt{e^x - 1} = t$.

③ $\int_{\sqrt{3}}^{\sqrt{7}} \frac{x^3\,dx}{\sqrt[3]{(x^2+1)^2}}$ uvođenjem smjene $z = x^2 + 1$.

④ $\int_{1}^{e} \frac{\sqrt[4]{1 + \ln x}}{x}\,dx$ uvođenjem smjene $t = 1 + \ln x$.

⑤ $\int_{-3}^{3} x^2 \sqrt{9 - x^2}\,dx$ uvođenjem smjene $x = 3\cos\varphi$

⑥ $\int_{5}^{1} \frac{t\,dt}{\sqrt{5 + 4t}}$ ⑦ $\int_{0}^{\frac{\pi}{4}} \frac{1 + tg^2\varphi}{1 + tg\,\varphi}\,d\varphi$ ⑧ $\int_{\ln 3}^{0} \frac{1 - e^x}{1 + e^x}\,dx$

⑨ $\int_{-1}^{0} \frac{dx}{1 + \sqrt[3]{x+1}}$ ⑩* $\int_{0}^{8} \sqrt{\frac{x}{6 - x}}\,dx$ ⑪* $\int_{0}^{\frac{\pi}{2}} \sin^3\varphi \sqrt{\cos\varphi}\,d\varphi$

Rješenja:

1. $\frac{1}{24}$ 2. $\frac{4-\pi}{2}$ 3. 3 4. $0{,}8(2\sqrt[4]{2} - 1)$ 5. $\frac{81\pi}{8}$ 6. $-\frac{17}{6}$

7. $\ln 2$ 8. $\ln\frac{4}{3}$ 9. $\frac{3}{2}(\ln 4 - 1)$ 10. $\frac{3(\pi - 2)}{2}$

11. $8/21$ (uvodimo smjenu $x = 6\sin^2 t$)

Izabrani **Zadaci za vježbu** sa rješenjima
(iz lekcije Računje određenih integrali i Smjena promjenjivih u određenim integralima)

①
$$\int_{\pi/6}^{\pi/3}\frac{dx}{\cos^2 x}=\operatorname{tg}x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\operatorname{tg}\frac{\pi}{3}-\operatorname{tg}\frac{\pi}{6}=\sqrt{3}-\frac{\sqrt{3}}{3}=\frac{2\sqrt{3}}{3}$$

②
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\sin x\,dx=-\cos x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}=-\left(\cos\frac{\pi}{3}-\cos\frac{\pi}{4}\right)=-\left(\frac{1}{2}-\frac{\sqrt{2}}{2}\right)=-\frac{1-\sqrt{2}}{2}$$

③
$$\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}\frac{dx}{\sqrt{1-x^2}}=\arcsin x\Big|_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}}=\arcsin\frac{\sqrt{3}}{2}-\arcsin\frac{1}{2}=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$$

④
$$\int_a^b x^m\,dx=\frac{x^{m+1}}{m+1}\Big|_a^b=\frac{1}{m+1}\left(b^{m+1}-a^{m+1}\right)$$

⑤
$$\int_0^1(e^x-1)^4e^x\,dx=\left|\begin{array}{l}e^x-1=t\\ e^x dx=dt\\ x=0\Rightarrow t=0\\ x=1\Rightarrow t=e-1\end{array}\right|=\int_0^{e-1}t^4\,dt=\frac{t^5}{5}\Big|_0^{e-1}=\frac{1}{5}(e-1)^5$$

⑥
$$\int_2^9\sqrt[3]{x-1}\,dx=\left|\begin{array}{l}x-1=t^3\\ dx=3t^2dt\\ x=2\Rightarrow t=1\\ x=9\Rightarrow t=2\end{array}\right|=\int_1^2\sqrt[3]{t^3}\cdot 3t^2\,dt=3\int_1^2 t^3\,dt=\frac{3}{4}t^4\Big|_1^2=$$
$$=\frac{3}{4}(16-1)=\frac{45}{4}$$

⑦
$$\int_0^{\ln 5}\frac{\sqrt{e^x-1}}{1+3e^{-x}}\,\overset{1\cdot e^x}{\underset{1\cdot e^x}{dx}}=\int_0^{\ln 5}\frac{e^x\sqrt{e^x-1}}{e^x+3}\,dx=\left|\begin{array}{l}e^x-1=t^2\\ e^x dx=2t\,dt\\ x=0\Rightarrow t=0\\ x=\ln 5\Rightarrow t=2\\ e^x=t^2+1\end{array}\right|=\int_0^2\frac{\sqrt{t^2}\cdot 2t}{t^2+1+3}\,dt$$
$$=2\int_0^2\frac{t^2+4-4}{t^2+4}\,dt=2\int_0^2 dt-2\int_0^2\frac{4}{t^2+4}\,dt=2t\Big|_0^2-8\cdot\frac{1}{2}\operatorname{arctg}\frac{t}{2}\Big|_0^2=$$
$$=4-4\left(\operatorname{arctg}1-\operatorname{arctg}0\right)=4-4\cdot\frac{\pi}{4}=4-\pi$$

Osobine određenih integrala

a) $\int_a^a f(x)\,dx=0$

b) $\int_a^b f(x)\,dx=-\int_b^a f(x)\,dx$

c) $\int_a^b c\cdot f(x)\,dx=c\int_a^b f(x)\,dx$

d) $\int_a^b[f(x)\pm g(x)]\,dx=\int_a^b f(x)\,dx\pm\int_a^b g(x)\,dx$

e) $\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx \quad \forall x$

⑧
$$\int_0^{\sqrt{7}}\frac{dx}{7+x^2}=\frac{1}{\sqrt{7}}\operatorname{arctg}\frac{x}{\sqrt{7}}=\frac{1}{\sqrt{7}}\left(\operatorname{arctg}\frac{\sqrt{7}}{\sqrt{7}}-\operatorname{arctg}\frac{0}{\sqrt{7}}\right)=\frac{1}{\sqrt{7}}\cdot\frac{\pi}{4}=\frac{\sqrt{7}\pi}{28}$$

⑨
$$\int_0^{1/2}\sqrt{1-x^2}\,dx=\left|\begin{array}{l}x=\sin t\\ dx=\cos t\,dt\\ x=0\Rightarrow\sin t=0\Rightarrow t=0\\ x=\frac{1}{2}\Rightarrow\sin t=\frac{1}{2}\Rightarrow t=\frac{\pi}{6}\end{array}\right|=\int_0^{\frac{\pi}{6}}\sqrt{1-\sin^2 t}\,\cos t\,dt=$$

kako je $\sin^2 t+\cos^2 t=1$

$$=\int_0^{\pi/6}\sqrt{\cos^2 t}\,\cos t\,dt=\int_0^{\pi/6}\cos^2 t\,dt=\frac{1}{2}\int_0^{\pi/6}(1+\cos 2t)\,dt=\frac{1}{2}t\Big|_0^{\frac{\pi}{6}}+\frac{1}{4}\sin 2t\Big|_0^{\frac{\pi}{6}}$$

$1=\cos^2 t+\sin^2 t$

$\cos 2t=\cos^2 t-\sin^2 t$

$$=\frac{1}{2}\cdot\frac{\pi}{6}+\frac{1}{4}\cdot\frac{\sqrt{3}}{2}=\frac{\pi}{12}+\frac{\sqrt{3}}{8}=\frac{2\pi+3\sqrt{3}}{24}$$

⑩ $\vee$
$$\int_0^4\frac{dx}{1+\sqrt{2x+1}}$$
uputa: smjena $2x+1=t^2$

rj. $2-\ln 2$

⑪ $\vee$
$$\int_0^1\frac{dx}{\sqrt{2-x^2+x}}$$
uputa: $-x^2+x+2=\ldots$

$\ldots=\frac{9}{4}-\left(x-\frac{1}{2}\right)^2$

$x-1=\frac{3}{2}t$

rj. $2\arcsin\frac{1}{3}$

⑫
$$\int_1^e x\ln x\,dx=\left|\begin{array}{ll}u=\ln x & dv=x\,dx\\ du=\frac{dx}{x} & v=\frac{x^2}{2}\end{array}\right|=\frac{1}{2}x^2\ln x\Big|_1^e-\frac{1}{2}\int_1^e x^2\cdot\frac{dx}{x}=$$
$$=\frac{1}{2}\left(e^2\ln e-1^2\ln 1\right)-\frac{1}{2}\int_1^e x\,dx=\frac{1}{2}e^2-\frac{1}{2}\cdot\frac{1}{2}x^2\Big|_1^e=$$
$$=\frac{1}{2}e^2-\frac{1}{4}\left(e^2-1\right)=\frac{1}{4}e^2+\frac{1}{4}$$

(#) Izračunati integral $\displaystyle\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5}\,dx$.

Rj. $x-4\sqrt{x}+5 = x-2\cdot\sqrt{x}\cdot 2+4+1 = (\sqrt{x}-2)^2+1$

$$\int_1^4 \frac{\sqrt{x}+2}{(\sqrt{x}-2)^2+1}\,dx = \left|\begin{array}{l} x=t^2 \\ dx=2t\,dt \\ x=1 \Rightarrow t=1 \\ x=4 \Rightarrow t=2 \end{array}\right| = \int_1^2 \frac{t+2}{(t-2)^2+1}\cdot 2t\,dt =$$

$$=2\int_1^2 \frac{t^2+2t}{t^2-4t+5}\,dt = 2\int_1^2 \frac{t^2+2t-6t+6t+5-5}{t^2-4t+5}\,dt = 2\int_1^2 \frac{t^2-4t+5}{t^2-4t+5}\,dt +$$

$$+2\int_1^2 \frac{6t-5}{t^2-4t+5}\,dt = 2\int_1^2 dt + 2\cdot 3\int_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5}$$

$$\int_1^2 \frac{2t-\frac{5}{3}}{t^2-4t+5}\,dt = \int_1^2 \frac{2t-4+4-\frac{5}{3}}{t^2-4t+5}\,dt = \int_1^2 \frac{2t-4}{t^2-4t+5}\,dt + \frac{7}{3}\int_1^2 \frac{dt}{t^2-4t+5}$$

$$\int_1^2 dt = t\Big|_1^2 = 2-1 = 1\ ,\quad \int_1^2 \frac{2t-4}{t^2-4t+5}\,dt = \left|\begin{array}{l} t^2-4t+5=s \\ (2t-4)dt=ds \\ t=1 \Rightarrow s=2 \\ t=2 \Rightarrow s=1 \end{array}\right| = \int_2^1 \frac{ds}{s} = \ln|s|\Big|_2^1$$

$$=\ln 1-\ln 2 = -\ln 2$$

$$\int_1^2 \frac{dt}{t^2-4t+5} = \int_1^2 \frac{dt}{(t-2)^2+1} = \left|\begin{array}{l} t-2=s \\ dt=ds \\ t=1 \Rightarrow s=-1 \\ t=2 \Rightarrow s=0 \end{array}\right| = \int_{-1}^0 \frac{ds}{s^2+1} = \operatorname{arctg} s\Big|_{-1}^0 = 0-\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$$

$$\int_1^4 \frac{\sqrt{x}+2}{x-4\sqrt{x}+5} = 2\cdot 1 + 6\left(-\ln 2 + \frac{7}{3}\cdot\frac{\pi}{4}\right) = 2-6\ln 2+\frac{7\pi}{2} \approx 8{,}8367$$

traženo rješenje

(#) Izračunati integral $\displaystyle\int_0^{\frac{\pi}{4}} \sin^5 x\,\cos^7 x\,dx$

Rj. $$\int_0^{\frac{\pi}{4}} \sin^5 x\cdot\cos^7 x\,dx = \int_0^{\frac{\pi}{4}} \sin^5 x\cdot\cos^6 x\cdot\cos x\,dx = \left|\begin{array}{l} \sin x=t \\ \cos x\,dx=dt \\ x=0 \Rightarrow t=0 \\ x=\frac{\pi}{4} \Rightarrow t=\frac{\sqrt{2}}{2} \\ \cos^6 x=(\cos^2 x)^3= \\ =(1-\sin^2 x)^3=(1-t^2)^3 \end{array}\right| =$$

$$=\int_0^{\frac{\sqrt{2}}{2}} t^5(1-t^2)^3\,dt =$$

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \end{array}$$

$$=\int_0^{\frac{\sqrt{2}}{2}} t^5(1-3t^2+3t^4-t^6)\,dt = \int_0^{\frac{\sqrt{2}}{2}} (t^5-3t^7+3t^9-t^{11})\,dt =$$

$$=\frac{1}{6}t^6\Big|_0^{\frac{\sqrt{2}}{2}} - \frac{3}{8}t^8\Big|_0^{\frac{\sqrt{2}}{2}} + \frac{3}{10}t^{10}\Big|_0^{\frac{\sqrt{2}}{2}} - \frac{1}{12}t^{12}\Big|_0^{\frac{\sqrt{2}}{2}} =$$

$$=\frac{1}{\underset{3}{\cancel{6}}}\cdot\frac{\overset{\overset{1}{\cancel{4}}}{\cancel{8}}}{\underset{16}{\cancel{64}}} - \frac{3}{\underset{1}{\cancel{8}}}\cdot\frac{\overset{\overset{1}{\cancel{2}}}{\cancel{16}}}{\underset{128}{\cancel{256}}} + \frac{3}{\underset{5}{\cancel{10}}}\cdot\frac{\overset{\overset{1}{\cancel{16}}}{\cancel{32}}}{\underset{64}{\cancel{1024}}} - \frac{1}{\underset{3}{\cancel{12}}}\cdot\frac{\overset{\overset{1}{\cancel{16}}}{\cancel{64}}}{\underset{256}{\cancel{4096}}} =$$

$$=\frac{1}{\underbrace{3\cdot 16}_{3\cdot 2^4}} - \frac{3}{\underbrace{128}_{2^7}} + \frac{3}{\underbrace{5\cdot 64}_{5\cdot 2^6}} - \frac{1}{\underbrace{3\cdot 256}_{3\cdot 2^8}} = \frac{5\cdot 2^4-3\cdot 3\cdot 5\cdot 2+3\cdot 3\cdot 2^2-5}{3\cdot 5\cdot 2^8}$$

$$=\frac{80-90+36-5}{3\cdot 5\cdot 2^8} = \frac{\overset{7}{\cancel{21}}}{\underset{1}{\cancel{3}}\cdot 5\cdot 2^8} = \frac{7}{1280}$$

traženo rješenje

Ⓣ Izračunati integral $\displaystyle\int_{-1}^{1} \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx$

Rj.

Metoda Ostrogradskog:

$$\int \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = (ax^2+bx+c)\sqrt{x^2+3} + \lambda \int \frac{dx}{\sqrt{x^2+3}} \quad \Big/ \frac{d}{dx}$$

$$\frac{2x^3-7x+4}{\sqrt{x^2+3}} = (2ax+b)\sqrt{x^2+3} + (ax^2+bx+c)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+3}} + \frac{\lambda}{\sqrt{x^2+3}} \Big/ \cdot$$

$$\Big/ \cdot \sqrt{x^2+3}$$

$$2x^3-7x+4 = (2ax+b)(x^2+3) + (ax^2+bx+c)x + \lambda$$

$$\underline{2x^3} - \underline{\underline{\underline{7x}}} + 4 = \underline{2ax^3} + \underline{\underline{bx^2}} + \underline{\underline{\underline{6ax}}} + 3b + \underline{ax^3} + \underline{\underline{bx^2}} + \underline{\underline{\underline{cx}}} + \lambda$$

x^3: $2a + a = 2 \Rightarrow 3a = 2$

$a = \frac{2}{3}$

x^2: $b + b = 0 \Rightarrow b = 0$

x: $6a + c = -7 \Rightarrow 6 \cdot \frac{2}{3} + c = -7$

$4 + c = -7$

$c = -11$

x^0: $3b + \lambda = 4$

$\lambda = 4$

Prema tome:

$$\int \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = \left(\frac{2}{3}x^2 - 11\right)\sqrt{x^2+3} + 4\int \frac{dx}{\sqrt{x^2+3}} =$$

$$= \frac{2}{3}x^2\sqrt{x^2+3} - 11\sqrt{x^2+3} + 4\ln\left|x+\sqrt{x^2+3}\right| + C$$

Prema tome

$$\int_{-1}^{1} \frac{2x^3-7x+4}{\sqrt{x^2+3}}\,dx = \frac{2}{3}x^2\sqrt{x^2+3}\,\Big|_{-1}^{1} - 11\sqrt{x^2+3}\,\Big|_{-1}^{1} + 4\ln\left|x+\sqrt{x^2+3}\right|\,\Big|_{-1}^{1} =$$

$$= \frac{2}{3}(2-2) - 11(2-2) + 4\left(\ln|1+2| - \ln|-1+2|\right) =$$

$$= 4(\ln 3 - \ln 1) = 4\ln 3$$ traženi rezultat

Ⓣ Izračunati: $\displaystyle\int_{3}^{4} \frac{6x+8}{x^2+x-6}\,dx$

Rj.

$$\frac{6x+8}{x^2+x-6} = \frac{A}{x-2} + \frac{B}{x+3} \quad \Big/ (x-2)(x+3)$$

$D = 1 + 24 = 25$

$x_{1,2} = \frac{-1 \pm 5}{2}$

$x_1 = -3 \quad x_2 = 2$

$x^2+x-6 = (x-2)(x+3)$

$6x + 8 = A(x+3) + B(x-2)$

$6x + 8 = (A+B)x + (3A-2B)$

$$\begin{aligned} A+B &= 6 \quad /\cdot 2 \\ 3A-2B &= 8 \\ \hline 2A+2B &= 12 \\ +\ 3A-2B &= 8 \\ \hline 5A &= 20 \\ A &= 4 \end{aligned}$$

$A + B = 6$

$4 + B = 6$

$B = 2$

$$\int \frac{6x+8}{x^2+x-6}\,dx = \int\left(\frac{4}{x-2} + \frac{2}{x+3}\right)dx = 4\int\frac{dx}{x-2} + 2\int\frac{dx}{x+3} =$$

$$= 4\ln|x-2| + 2\ln|x+3| + C$$

$$\int_{3}^{4} \frac{6x+8}{x^2+x-6}\,dx = 4\ln|x-2|\,\Big|_{3}^{4} + 2\ln|x+3|\,\Big|_{3}^{4} = 4(\ln 2 - \underbrace{\ln 1}_{=0}) +$$

$$+ 2(\ln 7 - \ln 6) = 4\ln 2 + 2\ln\frac{7}{6} = \ln 2^4 + \ln\left(\frac{7}{6}\right)^2$$

$$= \ln\frac{7^2}{2^2 \cdot 3^2} \cdot 2^4 = \ln\frac{49 \cdot 4}{9} = \ln\frac{196}{9}$$

$$\int_{3}^{4} \frac{6x+8}{x^2+x-6}\,dx = \ln\frac{196}{9}$$ traženo rješenje

(#) Izračunati integral $I=\int_0^1 \arcsin\frac{x}{2}\,dx$.

Rj.

$$\int \arcsin\frac{x}{2}\,dx = \left|\begin{matrix}\frac{x}{2}=t\\ \frac12 dx=dt\\ dx=2dt\end{matrix}\right| = 2\int \arcsin t\,dt = \left|\begin{matrix} u=\arcsin t & dv=dt\\ du=\frac{dt}{\sqrt{1-t^2}} & v=t\end{matrix}\right| =$$

$$=2\left(t\arcsin t - \int\frac{t}{\sqrt{1-t^2}}dt\right) = 2t\arcsin t - \int \frac{2t\,dt}{\sqrt{1-t^2}} \overset{(**)}{=}$$

$$\int\frac{-2t\,dt}{\sqrt{1-t^2}} = \left|\begin{matrix}1-t^2=s\\ -2t\,dt=ds\end{matrix}\right| = \int\frac{ds}{\sqrt{s}} = \int s^{-\frac12}ds = \frac{s^{\frac12}}{\frac12}+C=2\sqrt{s}+c =$$

$$=2\sqrt{1-t^2}+c$$

$$\overset{(**)}{=} 2t\arcsin t + 2\sqrt{1-t^2}+c = x\arcsin\frac{x}{2}+2\sqrt{1-\frac{x^2}{4}}+c$$

$$\int_0^1 \arcsin\frac{x}{2}\,dx = x\arcsin\frac{x}{2}\Big|_0^1 + 2\sqrt{1-\frac{x^2}{4}}\Big|_0^1 = \arcsin\frac12 + \left(2\sqrt{1-\frac14}-2\right) =$$

$$=\frac{\pi}{6}+\frac{2\sqrt3}{2}-2 = \frac{\pi}{6}+\sqrt3-2$$

Nepravi integrali

Nepravi integral u granicama od a do $+\infty$ je oblika

$$I=\int_a^{+\infty} f(x)\,dx$$

Rješavamo ga na sljedeći način:

$$\int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty}\int_a^b f(x)\,dx$$

Ako postoji

$$\lim_{b\to\infty}\int_a f(x)\,dx\,.$$

kažemo da integral konvergira ili da postoji nepravi integral, a ako limes ne postoji (kao realan broj), kažemo da integral divergira ili da nepravi integral ne postoji.

① Izračunati:

a) $$\int_1^{+\infty}\frac{dx}{x^4} = \lim_{a\to+\infty}\int_1^a\frac{dx}{x^4} = \lim_{a\to+\infty}\int_1^a x^{-4}dx = \lim_{a\to+\infty}\frac{x^{-3}}{-3}\Big|_1^a =$$

$$= -\frac13\lim_{a\to\infty}\frac{1}{x^3}\Big|_1^a = -\frac13\lim_{a\to\infty}\left(\overset{0}{\cancel{\frac{1}{a^3}}}-1\right) = \left(-\frac13\right)(-1) = \frac13$$

b) $$\int_1^{+\infty}\frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_1^b\frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\int_1^b x^{-\frac12}dx = \lim_{b\to\infty}\frac{x^{\frac12}}{\frac12}\Big|_1^b = 2\lim\sqrt{x}\Big|_1^b$$

$$= 2\lim_{b\to\infty}\left(\sqrt{b}-1\right) = +\infty,$$ integral divergira

c) $$\int_0^{+\infty} e^{-ax}dx = \lim_{t\to\infty}\int_0^t e^{-ax}dx = \left|\begin{matrix}-ax=s & x=0 \Rightarrow s=0\\ -a\,dx=ds & x=t \Rightarrow s=-at\\ dx=-\frac1a ds & \end{matrix}\right| =$$

$(a>0)$

$$= \lim_{t\to\infty}\int_0^{-at} e^s\left(-\frac1a\right)ds = -\frac1a\lim_{t\to\infty} e^s\Big|_0^{-at} = -\frac1a\lim_{t\to\infty}\left(\overset{0}{\cancel{e^{-at}}}-1\right) = \frac1a$$

d)^v $\int_2^{+\infty}\frac{\ln x}{x}\,dx$ Rj. divergira. e)^v $\int_1^{+\infty}\frac{dx}{x^2(x+1)}$ Rj. $1-\ln 2$ f)^v $\int_0^{+\infty} xe^{-x^2}dx$ Rj. $\frac12$

Zadaci za vježbu

Integrali s beskonačnim granicama

U zadacima 2366 2385 izračunati donje nesvojstvene integrale (ili ustanoviti njihovu divergenciju).

2366. $\int_1^\infty \frac{dx}{x^4}$. **2367.** $\int_1^\infty \frac{dx}{\sqrt{x}}$. **2368.** $\int_0^\infty e^{-ax}\,dx \quad (a>0)$.

2369. $\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$. **2370.** $\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}$. **2371.** $\int_2^\infty \frac{\ln x}{x}\,dx$.

2372. $\int_1^\infty \frac{dx}{x^2(x+1)}$. **2373.** $\int_0^\infty \frac{x}{(1+x)^3}\,dx$ **2374.** $\int_{\sqrt{2}}^\infty \frac{dx}{x\sqrt{x^2-1}}$.

2375. $\int_{a^2}^\infty \frac{dx}{x\sqrt{1+x^2}}$. **2376.** $\int_0^\infty xe^{-x^2}\,dx$. **2377.** $\int_0^\infty x^3 e^{-x^2}\,dx$.

2378. $\int_0^\infty x\sin x\,dx$. **2379.** $\int_0^\infty e^{-\sqrt{x}}\,dx$. **2380.** $\int_0^\infty e^{-x}\sin x\,dx$.

2381. $\int_0^\infty e^{-ax}\cos bx\,dx$. **2382.** $\int_1^\infty \frac{\operatorname{arctg} x}{x^2}\,dx$. **2383.** $\int_0^\infty \frac{dx}{1+x^3}$.

2384. $\int_{-\infty}^\infty \frac{dx}{(x^2+1)^2}$. **2385** $\int_1^\infty \frac{\sqrt{x}}{(1+x)^2}\,dx$.

U zadacima 2386 — 2393 ispitati da li su dati integrali konvergentni.

2386. $\int_0^\infty \frac{x}{x^3+1}\,dx$. **2387.** $\int_1^\infty \frac{x^3+1}{x^4}\,dx$.

2388. $\int_0^\infty \frac{x^{13}}{(x^5+x^3+1)^3}\,dx$. **2389.** $\int_1^\infty \frac{\ln(x^2+1)}{x}\,dx$.

2390. $\int_0^\infty \sqrt{xe^{-x}}\,dx$. **2391.** $\int_0^\infty \frac{x\operatorname{arctg} x}{\sqrt[3]{1+x^4}}\,dx$.

2392. $\int_{e^2}^\infty \frac{dx}{x\ln\ln x}$. **2393.** $\int_e^\infty \frac{dx}{x(\ln x)^{\frac{3}{2}}}$.

Rješenja

2366. $\frac{1}{3}$.

2367. Divergira. **2368.** $\frac{1}{a}$.

2369. Divergira. **2370.** π.

2371. Divergira. **2372.** $1-\ln 2$.

2373. $\frac{1}{2}$. **2374.** $\frac{\pi}{4}$.

2375. $\ln\frac{\sqrt{a^4+1}+1}{a^2}$. **2376.** $\frac{1}{2}$.

2377. $\frac{1}{2}$. **2378.** Divergira.

2379. 2. **2380.** $\frac{1}{2}$.

2381. $\frac{a}{a^2+b^2}$, ako je $a>0$,
a ne postoji ako je $a<0$.

2382. $\frac{\pi}{4}+\frac{1}{2}\ln 2$. **2383.** $\frac{2\pi}{3\sqrt{3}}$.

2384. $\frac{\pi}{2}$. **2385.** $\frac{1}{2}+\frac{\pi}{4}$.

2386. Konvergira. **2387.** Divergira.

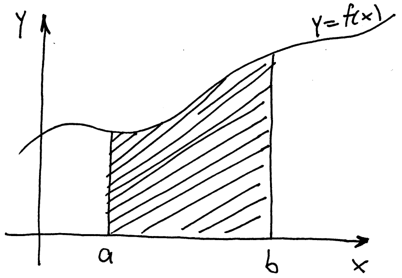
2388. Konvergira. **2389.** Divergira.

2390. Konvergira. **2391.** Divergira.

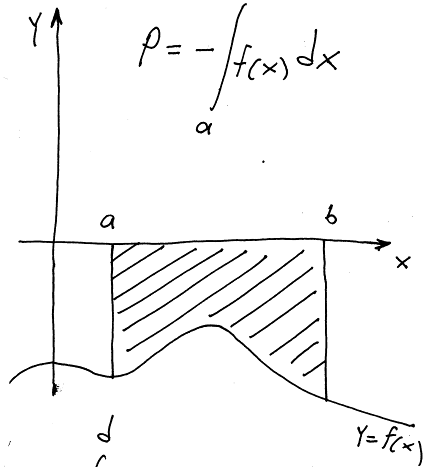
2392. Divergira. **2393.** Konvergira.

Primjena određenog integrala

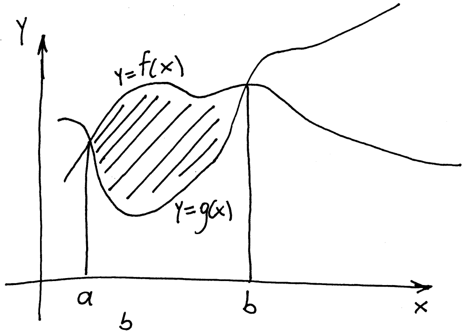
I Izračunavanje površine ravne figure



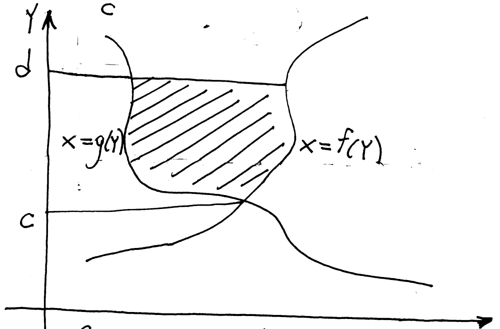
$$P=\int_a^b f(x)\,dx$$



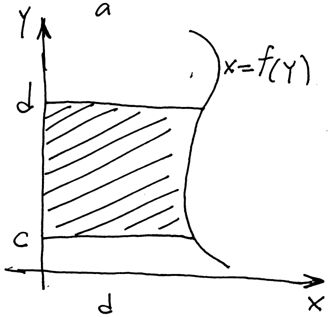
$$P=-\int_a^b f(x)\,dx$$



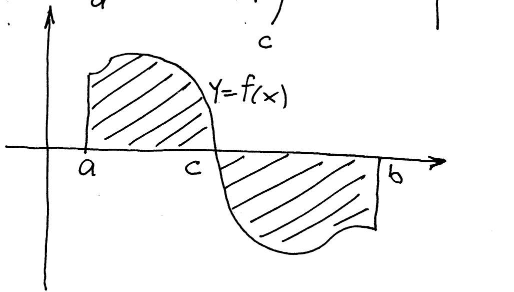
$$P=\int_a^b [f(x)-g(x)]\,dx$$



$$P=\int_c^d [f(Y)-g(Y)]\,dY$$



$$P=\int_c^d f(Y)\,dY$$



$$P=\int_a^c f(x)\,dx+\left|\int_c^b f(x)\,dx\right|$$

① Izračunati površinu ravne figure koja je ograničena linijama $Y=4-(x-2)^2$ i $Y=0$.

Rj.

$Y=4-(x-2)^2$

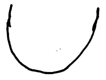
$Y=4-(x^2-4x+4)$

$Y=-x^2+4x$

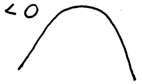
$Y=-x(x-4)$

Kriva $Y=ax^2+bx+c$ ima grafik u obliku parabole. Tjeme parabole $T\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$

za $a>0$



za $a<0$

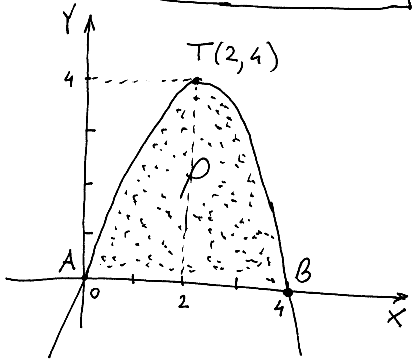


Nule $A(0,0)$ i $B(4,0)$

$$-\frac{b}{2a}=-\frac{4}{2\cdot(-1)}=2$$

$$\frac{4ac-b^2}{4a}=\frac{0-16}{-4}=4$$

Tjeme parabole $Y=4-(x-2)^2$ je u tački $(2,4)$.



$$P=\int_0^4(-x^2+4x)\,dx=\int_0^4(-x^2)\,dx+\int_0^4 4x\,dx=-\frac{x^3}{3}\Big|_0^4+4\cdot\frac{x^2}{2}\Big|_0^4=-\frac{1}{3}(4^3-0^3)$$

$$+2(4^2-0^2)=-\frac{1}{3}\cdot 64+32=\frac{96}{3}-\frac{64}{3}=\frac{32}{3}$$

② Izračunati površinu ravne figure koja je ograničena krivom $Y=x^2-4x+3$ i pravama $Y=0$, $x=0$ i $x=2$.

Rj. $Y=x^2-4x+3$

$D=16-12=4$

$x_{1,2}=\frac{4\pm 2}{2}$

Nule krive

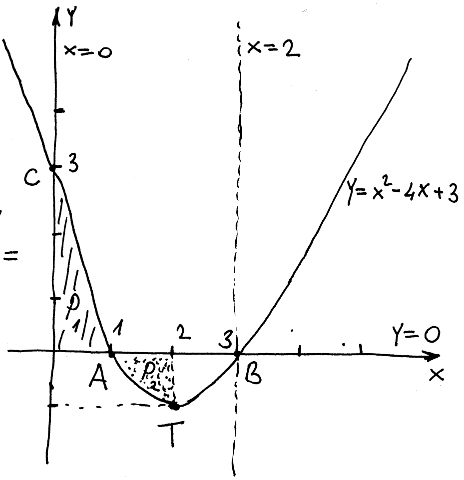
$A(1,0)$ i $B(3,0)$

$$-\frac{b}{2a}=-\frac{-4}{2}=2$$

$$\frac{4ac-b^2}{4a}=\frac{12-16}{4}=-1$$

Tjeme krive $Y=x^2-4x+3$ je u tački $T(2,-1)$.

$C(0,3)$ je tačka presjeka krive sa Y-osom



$P=P_1+P_2$

$$P_1=\int_0^1(x^2-4x+3)\,dx=\frac{x^3}{3}\Big|_0^1-4\cdot\frac{x^2}{2}\Big|_0^1+3x\Big|_0^1=$$

$$=\frac{1}{3}(1-0)-2(1-0)+3(1-0)=$$

$$=\frac{1}{3}+1=\frac{4}{3}$$

$$P_2=-\int_1^2(x^2-4x+3)\,dx=-\left(\frac{x^3}{3}\Big|_1^2-4\cdot\frac{x^2}{2}\Big|_1^2+3x\Big|_1^2\right)=-\left(\frac{1}{3}(8-1)-2(4-1)+3\cdot 1\right)$$

$$=-\left(\frac{7}{3}-6+3\right)=-\left(-\frac{2}{3}\right)=\frac{2}{3}$$

$P=\frac{4}{3}+\frac{2}{3}=2$ tražena površina ravne figure.

③ Izračunati površinu ravne figure kojeg čine parabola $Y=x^2-2x+2$ i prava $x+2Y-9=0$.

Rj.

prava $x+2Y-9=0$

$2Y=-x+9$

$Y=-\frac{1}{2}x+\frac{9}{2}$

prava prolazi kroz tačke $A\left(0,\frac{9}{2}\right)$ i $B(9,0)$.

$$-\frac{b}{2a}=-\frac{-2}{2}=1$$

$$\frac{4ac-b^2}{4a}=\frac{8-4}{4}=1$$

$Y=x^2-2x+2$

$D=4-8=-4<0$

kriva ne siječe x-o
-nema nula

$x=0 \Rightarrow Y=2$

$C(0,2)$ je presjek krive sa Y-osom

$T(1,1)$ je tjeme parabole

Trebamo naći još tačke presjeka prave i parabole.

$$\frac{\begin{array}{l} y = x^2-2x+2 \\ x+2y-9=0 \end{array}}{\begin{array}{l} y = x^2-2x+2 \\ x=-2y+9 \end{array}}$$

$$y=(-2y+9)^2-2(-2y+9)+2$$

$$y=4y^2\underline{-36y}+81\underline{+4y}-18+2$$

$$4y^2-33y+65=0$$

$$D=33^2-16\cdot 65=49$$

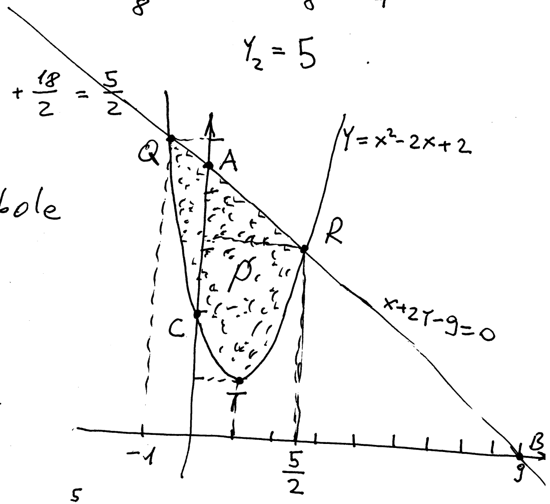
$$y_{1,2}=\frac{33\pm 7}{8} \qquad y_1=\frac{26}{8}=\frac{13}{4}$$

$$y_2=5$$

$$y_1=\frac{13}{4} \Rightarrow x=-2\cdot\frac{13}{4}+9=-\frac{13}{2}+\frac{18}{2}=\frac{5}{2}$$

$$y_2=5 \Rightarrow x=-2\cdot 5+9=-1$$

Tačke presjeka prave i parabole su $R(\frac{5}{2},\frac{13}{4})$ i $Q(-1,5)$



$$P=\int_{-1}^{\frac{5}{2}}\left[\left(-\frac{1}{2}x+\frac{9}{2}\right)-\left(x^2-2x+2\right)\right]dx$$

$$\int_{-1}^{\frac{5}{2}}\left(-\frac{1}{2}x+\frac{9}{2}\right)dx=-\frac{1}{2}\cdot\frac{x^2}{2}\Big|_{-1}^{\frac{5}{2}}+\frac{9}{2}x\Big|_{-1}^{\frac{5}{2}}=-\frac{1}{4}\left(\frac{25}{4}-1\right)+\frac{9}{2}\left(\frac{5}{2}-(-1)\right)$$

$$=-\frac{1}{4}\cdot\frac{21}{4}+\frac{9}{2}\cdot\frac{7}{2}=\frac{231}{16}$$

$$\int_{-1}^{\frac{5}{2}}(x^2-2x+2)\,dx=\frac{x^3}{3}\Big|_{-1}^{\frac{5}{2}}-2\frac{x^2}{2}\Big|_{-1}^{\frac{5}{2}}+2x\Big|_{-1}^{\frac{5}{2}}=\frac{1}{3}\left(\frac{125}{8}-(-1)\right)-\left(\frac{25}{4}-1\right)+$$

$$+2\left(\frac{5}{2}-(-1)\right)=\frac{1}{3}\cdot\frac{133}{8}-\frac{21}{4}+2\cdot\frac{7}{2}=\frac{133}{24}+\frac{7}{4}=\frac{175}{24}$$

$$P=\frac{231}{16}-\frac{175}{24}=\frac{231}{4\cdot 4}-\frac{175}{6\cdot 4}=\frac{686}{96}=\frac{343}{48}$$

(4.) Izračunati površinu ravne figure koja je ograničena krivom $y^2=2x+1$ i pravom $y=2x-1$.

Rj. prava $y=2x-1$ prolazi kroz tačke $A(0,-1)$ i $B(\frac{1}{2},0)$.

$$y^2=2x+1$$

$$2x=y^2-1$$

$$x=\frac{1}{2}y^2-\frac{1}{2}$$

$x=0 \Rightarrow y^2=1$

$A(0,-1)$ i $B(0,1)$ su tačke presjeka f-je sa y-osom

$C(-\frac{1}{2},0)$ je nula f-je

$$D=1 \qquad -\frac{D}{4a}=-\frac{1}{4\cdot\frac{1}{2}}$$

$$-\frac{b}{2a}=-\frac{0}{2\cdot\frac{1}{2}}=0$$

$T(-\frac{1}{2},0)$ je tjeme parabole

Kriva oblika $x=ay^2+bx+c$ ima grafik u obliku parabole

⊃ ili ⊂

$a<0$ $\qquad$ $a>0$

Tjeme krive se traži po formuli $T(-\frac{D}{4a},-\frac{b}{2a})$

Tražimo još tačke presjeka krive i prave

$$\frac{\begin{array}{l} y=2x-1 \\ y^2=2x+1 \end{array}}{(2x-1)^2=2x+1}$$

za $x=0$ $\Downarrow$ $y=-1$

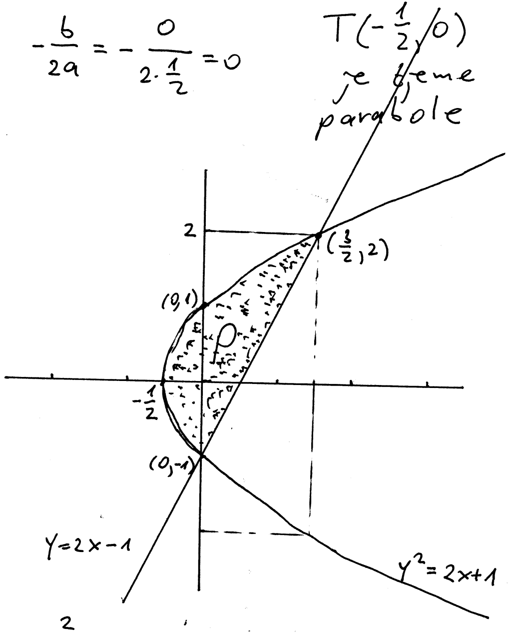
$$4x^2-4x+1-2x-1=0$$

$$4x^2-6x=0$$

$$2x(2x-3)=0$$

$$x=0 \lor x=\frac{3}{2}$$

$D(0,-1)$ i $E(\frac{3}{2},2)$ su tačke presjeka krive i prave



$$y=2x-1 \Rightarrow x=\frac{1}{2}y+\frac{1}{2}$$

$$y^2=2x+1 \Rightarrow x=\frac{1}{2}y^2-\frac{1}{2}$$

$$P=\int_{-1}^{2}\left[\frac{1}{2}y+\frac{1}{2}-\left(\frac{1}{2}y^2-\frac{1}{2}\right)\right]dy=\frac{1}{2}\int_{-1}^{2}(y+1-y^2+1)\,dy=\frac{1}{2}\int_{-1}^{2}(-y^2+y+2)\,dy=$$

$$=\frac{1}{2}\cdot\left[\left(-\frac{y^3}{3}\right)\Big|_{-1}^{2}+\frac{y^2}{2}\Big|_{-1}^{2}+2y\Big|_{-1}^{2}\right]=\frac{1}{2}\left[-\frac{1}{3}(8+1)+\frac{1}{2}(4-1)+2(2+1)\right]=$$

$$=\frac{1}{2}\left(-3+\frac{3}{2}+6\right)=\frac{1}{2}\cdot\frac{9}{2}=\frac{9}{4}$$ tražena površina

(#) Na parabolu $Y=1-x^2$ povučena je normala u tački presjeka parabole i pozitivnog dijela x-ose. Odrediti površinu figure koju čine data parabola, povučena normala i Y-osa.

Rj.

$Y=1-x^2$

$Y(0)=1$

(0,1) je presjek sa Y-osom

$1-x^2=0$

$x^2=1$

$x_{1,2}=\pm 1$

(-1,0) i (1,0) su nule f-je

$Y=-x^2+1$ parabola izgleda ∩

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

$-\frac{b}{2a}=-\frac{0}{2\cdot(-1)}=0$

$D=0-4(-1)(1)=4$

$-\frac{D}{4a}=-\frac{4}{4\cdot(-1)}=1$

$T(0,1)$

$Y-Y_1=Y'(x_1)(x-x_1)$ jednačina tangente u tački (x_1,Y_1)

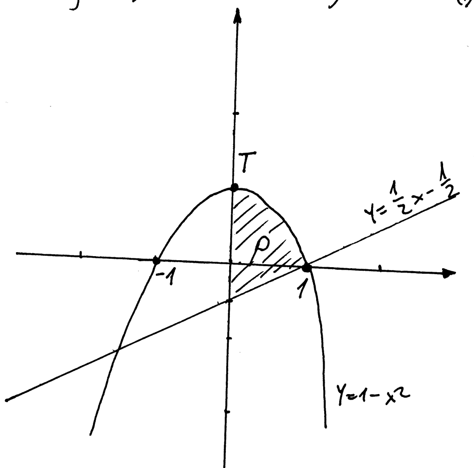
$Y-Y_1=-\frac{1}{Y'(x_1)}(x-x_1)$ jednačina normale u tački (x_1,Y_1)

$Y'=-2x$ presjek parabole i pozitivnog dijela x-ose je tačka (1,0)

$Y'(1)=-2$

$Y-0=-\frac{1}{-2}(x-1)$

$Y=\frac{1}{2}x-\frac{1}{2}$ jednačina normale u tački (1,0)



$$P=\int_0^1\left[(1-x^2)-\left(\frac{1}{2}x-\frac{1}{2}\right)\right]dx=$$

$$=\int_0^1\left(-x^2-\frac{1}{2}x+\frac{3}{2}\right)dx=-\frac{1}{3}x^3\Big|_0^1-\frac{1}{4}x^2\Big|_0^1+\frac{3}{2}x\Big|_0^1$$

$$=-\frac{1\cdot 4}{3\cdot 4}-\frac{1\cdot 3}{4\cdot 3}+\frac{3}{2}=\frac{3\cdot 6}{2\cdot 6}-\frac{7}{12}=\frac{18-7}{12}=\frac{11}{12}$$

$P=\frac{11}{12}$ tražena površina

(#) Izračunati površinu koju gradi kriva $Y=x^2+x-6$ zajedno sa svojim tangentama povučenim na tu krivu u nultačkama krive.

Rj.

$Y=x^2+x-6$

$D=1+24=25$

$Y=(x-2)(x+3)$

$x_1=2 \quad x_2=-3$

(2,0) i (-3,0) su nule f-je

$f(0)=-6$

(0,-6) je tačka presjeka f-je sa Y-osom

$T\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$ je tjeme f-je

$a>0$ f-ja je ∪ oblika

$-\frac{b}{2a}=-\frac{1}{2}$, $-\frac{D}{4a}=-\frac{25}{4}=-6\frac{1}{4}$

$T\left(-\frac{1}{2}, -6\frac{1}{4}\right)$

$Y-Y_1=k(x-x_1)$ jednačina prave kroz tačku (x_1,Y_1) i koeficijentom k

u slučaju tangente $k=Y'(x_1)$

$Y'=2x+1$

$(-3,0)$, $Y'(-3)=-5$

$Y-0=-5(x+3)$

$Y=-5x-15$ jednačina tangente na krivu Y u tački (-3,0)

$(2,0)$, $Y'(2)=5$

$Y-0=5(x-2)$

$Y=5x-10$ jednačina tangente na krivu Y u tački (2,0)

presjek pravih:

$Y=-5x-15$ (1)

$Y=5x-10$ (2)

(1)+(2): $2Y=-25$

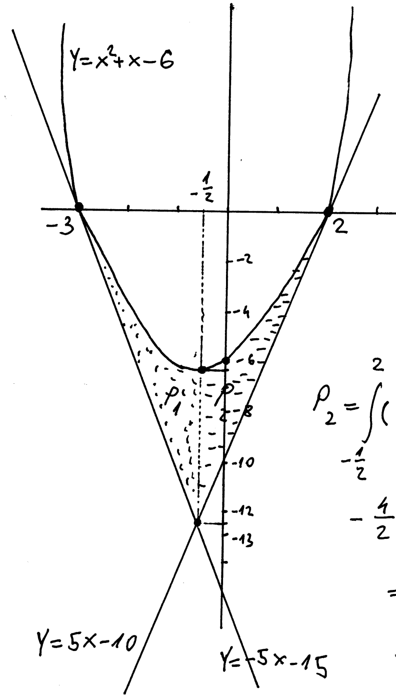
$Y=-\frac{25}{2}=-12\frac{1}{2}$

(1)-(2): $-10x-5=0$

$-10x=5$

$x=-\frac{1}{2}$

$\left(-\frac{1}{2}, -12\frac{1}{2}\right)$ je tačka presjeka pravih



$P=P_1+P_2$

$$P_1=\int_{-3}^{-\frac{1}{2}}\left(x^2+x-6-(-5x-15)\right)dx=\int_{-3}^{-\frac{1}{2}}(x^2+6x+9)dx=$$

$$=\frac{1}{3}x^3\Big|_{-3}^{-\frac{1}{2}}+\frac{6}{2}x^2\Big|_{-3}^{-\frac{1}{2}}+9x\Big|_{-3}^{-\frac{1}{2}}=\frac{1}{3}\left(-\frac{1}{8}+27\right)$$

$$+3\left(\frac{1}{4}-9\right)+9\left(-\frac{1}{2}+3\right)=\frac{1}{3}\cdot\frac{215}{8}+3\cdot\frac{-35}{4}$$

$$+9\cdot\frac{5}{2}=\frac{215}{24}-\frac{630}{24}+\frac{540}{24}=\frac{125}{24}$$

$$P_2=\int_{-\frac{1}{2}}^{2}\left(x^2+x-6-(5x-10)\right)dx=\int_{-\frac{1}{2}}^{2}(x^2-4x+4)dx=\frac{1}{3}x^3\Big|_{-\frac{1}{2}}^{2}-$$

$$-\frac{4}{2}x^2\Big|_{-\frac{1}{2}}^{2}+4x\Big|_{-\frac{1}{2}}^{2}=\frac{1}{3}\left(8+\frac{1}{8}\right)-2\left(4-\frac{1}{4}\right)+4\left(2+\frac{1}{2}\right)=$$

$$=\frac{1}{3}\cdot\frac{65}{8}-2\cdot\frac{15}{4}+4\cdot\frac{5}{2}=\frac{65}{24}-\frac{180}{24}+\frac{240}{24}=\frac{125}{24}$$

$P=P_1+P_2=\frac{125}{24}+\frac{125}{24}=\frac{125}{12}$ tražena površina

Ⓘ# Izračunati površinu figure koju ograničavaju linije $x = y^2 - 2y - 3$; $y = 3 - 3x$

$y = 3 - 3x$
$3x = 3 - y$
$x = 1 - \frac{1}{3}y$

Rj. Nađimo presječnu tačku ovih linija

$$x = y^2 - 2y - 3$$
$$y = 3 - 3x$$

$$x = (3-3x)^2 - 2(3-3x) - 3$$
$$x = 9 - 18x + 9x^2 - 6 + 6x - 3$$
$$9x^2 - 13x = 0$$
$$x(9x - 13) = 0$$
$$x = 0 \quad \text{ili} \quad 9x = 13$$
$$x = \frac{13}{9}$$

$x = 0 \Rightarrow y = 3$

$x = \frac{13}{9} \Rightarrow y = 3 - \cancel{3}\cdot\frac{13}{\cancel{9}_3} = \frac{9}{3} - \frac{13}{3} = -\frac{4}{3}$

$A(0,3)$; $B(\frac{13}{9}, -\frac{4}{3})$ su presječne tačke linija

$x = y^2 - 2y - 3$
$y = 0 \Rightarrow x = -3$
$(-3,0)$ je presjek krive sa x-osom

$x = y^2 - 2y - 3$ je kriva oblika parabole $\subset$ čije je tjeme $T(-\frac{D}{4a}, -\frac{b}{2a})$

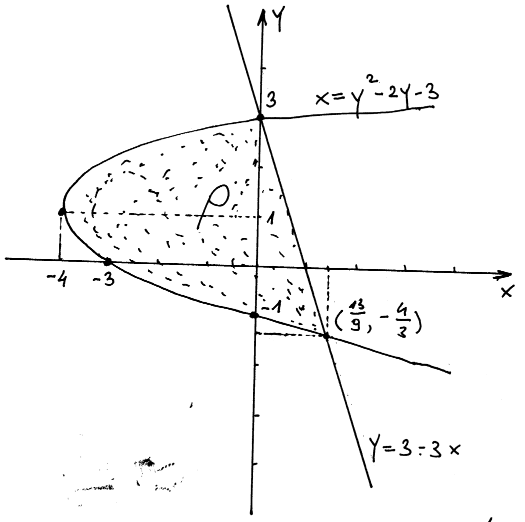
$-\frac{b}{2a} = -\frac{-2}{2} = 1$, $D = 4 + 12 = 16$ $-\frac{D}{4a} = -\frac{16}{4} = -4$

$T(1, -4)$

$y_{1,2} = \frac{2 \pm 4}{2}$

$y_1 = \frac{-2}{2} = -1$ $y_2 = \frac{6}{2} = 3$

$M_1(0,-1)$; $M_2(0,3)$ su presjek parabole sa y-osom



$$P = \int_{-\frac{4}{3}}^{3} \left[\left(1 - \frac{1}{3}y\right) - (y^2 - 2y - 3)\right] dy =$$

$$= \int_{-\frac{4}{3}}^{3} \left(-y^2 + \frac{5}{3}y + 4\right) dy =$$

$$= -\frac{1}{3}y^3\Big|_{-\frac{4}{3}}^{3} + \frac{5}{3}\cdot\frac{1}{2}y^2\Big|_{-\frac{4}{3}}^{3} + 4y\Big|_{-\frac{4}{3}}^{3} =$$

$$= -\frac{1}{3}\left(27 + \frac{64}{27}\right) + \frac{5}{6}\left(9 - \frac{16}{9}\right) + 4\left(3 + \frac{4}{3}\right)$$

$$= -\frac{1}{3}\cdot\frac{793}{27} + \frac{5}{6}\cdot\frac{65}{9} + 4\cdot\frac{13}{3} =$$

$$= -\frac{793}{81} + \frac{325}{54} + \frac{52}{3} = \frac{-793\cdot 2 + 325\cdot 3 + 52\cdot 54}{162} = \frac{-1586 + 975 + 2808}{162}$$

$$P = \frac{2197}{162} = 13\frac{91}{162}$$. tražena površina

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Izračunati površinu ravne figure koja je ograničena parabolama $y = -x^2 - 4x$; $y = x^2 + 2x$.

Rj.

Za parabolu $y = -x^2 - 4x$ znamo da je $\cap$ oblika. Vidimo da x-osu siječe u tačkama -4 i 0.

$y' = -2x - 4$
$-2x - 4 = 0$
$x = -2$

Tjeme ove parabole je $T(-2, 4)$

Za parabolu $y = x^2 + 2x$ znamo da je $\cup$ oblika. Vidimo da x-osu siječe u tačkama -2 i 0.

$y' = 2x + 2$
$2x + 2 = 0$
$x = -1$

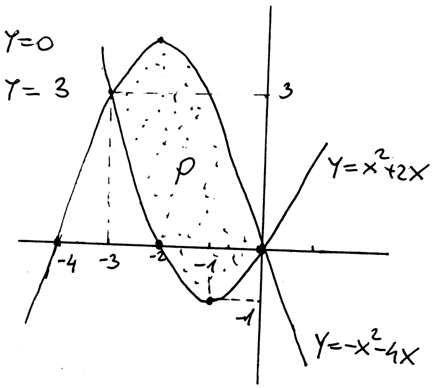
Tjeme ove parabole je $T(-1, 1)$

Pronađimo još presječne tačke dvije date parabole.

$$y = -x^2 - 4x$$
$$y = x^2 + 2x$$

$$-2x^2 - 6x = 0 \quad /:(-2)$$
$$x^2 + 3x = 0$$
$$x(x+3) = 0$$

$x = 0 \Rightarrow y = 0$
$x = -3 \Rightarrow y = 3$



$$P = \int_{-3}^{0} \left[(-x^2 - 4x) - (x^2 + 2x)\right] dx = \int_{-3}^{0} (-2x^2 - 6x)\, dx = -\frac{2}{3}x^3\Big|_{-3}^{0} - 6\cdot\frac{1}{2}x^2\Big|_{-3}^{0} =$$

$$= -\frac{2}{3}(0 - (-27)) - 3(0 - 9) = -18 + 27 = 9$$ vrijednost tražene površine

(#) Izračunati površinu ravne figure koja je ograničena krivim linijama $x=y^2-1$ i $x=-y^2-2y+3$

Rj: Za krivu $x=y^2-1$ vidimo da je sljedećeg oblika C
y-osu siječe u tačkama -1 i 1

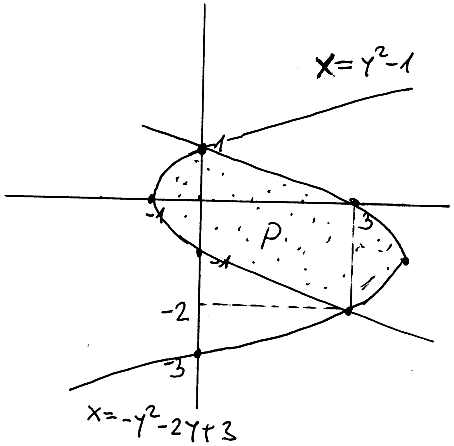
Kriva $x=-y^2-2y+3$ je oblika Ɔ. y-osu siječe u tačkama -3 i 1.

Pronađimo presječne tačke dvije date krive.

$$\begin{array}{r} x=y^2-1 \\ -\quad x=-y^2-2y+3 \\ \hline 2y^2+2y-4=0 \quad /:2 \\ y^2+y-2=0 \\ (y+2)(y-1)=0 \end{array}$$

$y=-2 \Rightarrow x=3$

$y=1 \Rightarrow x=0$



$$P=\int_{-2}^{1}\left[(-y^2-2y+3)-(y^2-1)\right]dy=$$

$$=\int_{-2}^{1}(-2y^2-2y+4)\,dy=-\frac{2}{3}y^3\Big|_{-2}^{1}-y^2\Big|_{-2}^{1}+4y\Big|_{-2}^{1}=$$

$24\cdot3=72$

$$=-\frac{2}{3}(1-(-8))-(1-4)+4(1-(-2))=$$

$=-6+3+12=9$ tražena površina

(#) Izračunati površinu ravne figure koja je ograničena parabolama $x=y^2-4y+3$ i $x=-y^2+2y+3$.

Rj: Parabola $x=y^2-4y+3$ je C oblika. $x=(y-3)(y-1)$
y-osu siječe u tačkama 1 i 3.

$x'=2y-4$

$2y-4=0$

$y=2$

$x=4-8+3=-1$ Tjeme ove parabole je $T(-1,2)$

Parabola $x=-y^2+2y+3$ je Ɔ oblika. $x=-(y+1)(y-3)$
y-osu siječe u tačkama -1 i 3.

$x'=-2y+2$

$-2y+2=0$

$y=1$

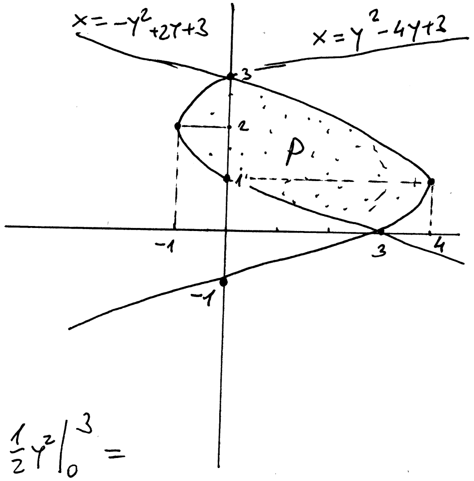
$x=-1+2+3=4$ Tjeme ove parabole je $T(4,1)$

Pronađimo još presječne tačke dvije date parabole

$$\begin{array}{r} x=y^2-4y+3 \\ -\quad x=-y^2+2y+3 \\ \hline 2y^2-6y=0 \\ 2y(y-3)=0 \end{array}$$

$y=0 \Rightarrow x=3$

$y=3 \Rightarrow x=0$



$$P=\int_{0}^{3}\left[(-y^2+2y+3)-(y^2-4y+3)\right]dy=$$

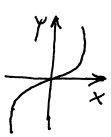
$$=\int_{0}^{3}(-2y^2+6y)\,dy=-\frac{2}{3}y^3\Big|_{0}^{3}+6\cdot\frac{1}{2}y^2\Big|_{0}^{3}=$$

$=-2\cdot9+3\cdot9=9$ tražena površina

#) Izračunati površinu figure koja je određena linijama $Y=-x$, $Y=\sqrt[3]{x}$, $Y=3x-2$.

Rj. Grafički nije teško predstaviti prave $Y=-x$ i $Y=3x-2$. Problem predstavlja kriva $Y=\sqrt[3]{x}$.

Ako znamo da kriva $Y=x^3$ izgleda ovako



Onda nije teško nacrtati krivu $x=Y^3$ što je ekvivalentno sa $Y=\sqrt[3]{x}$.

Pronađimo tačke presjeka datih krivih.

$$\frac{\begin{array}{l} Y=-x \\ Y=3x-2 \end{array}}{-x=3x-2}$$
$$-4x=-2$$
$$x=\frac{1}{2} \Rightarrow Y=-\frac{1}{2}$$

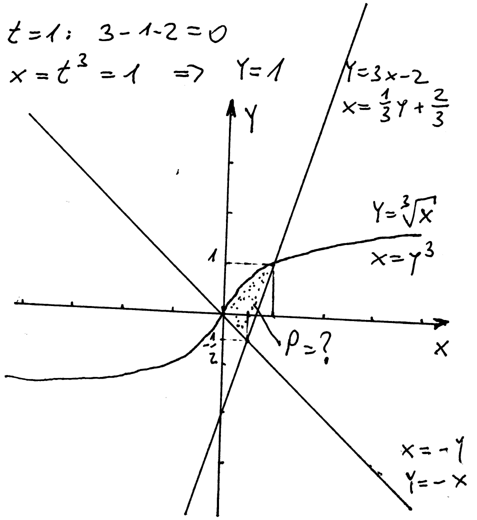
$$\frac{\begin{array}{l} Y=-x \\ Y=\sqrt[3]{x} \end{array}}{Y=-x}$$
$$\frac{Y^3=x}{-x^3=x}$$
$$x^3+x=0$$
$$x(x^2+1)=0$$
$$x=0 \Rightarrow Y=0$$

$$\frac{\begin{array}{l} Y=3x-2 \\ Y=\sqrt[3]{x} \end{array}}{\sqrt[3]{x}=3x-2}$$
$$(3x-2)^3=x$$
$$27x^3-3\cdot(3x)^2\cdot 2+$$
$$+3\cdot 3x\cdot(-2)^2+(-2)^3=x$$
$$27x^3-54x^2+36x-8=x$$
$$27x^3-54x^2+35x-8=0$$

pokušajmo riješiti sistem na drugi način

$$\sqrt[3]{x}=3x-2$$
$$x=t^3$$
$$3t^3-2=t$$
$$3t^3-t-2=0$$
$$t=1:\ 3-1-2=0$$
$$x=t^3=1 \Rightarrow Y=1$$

$$\boxed{3x=Y+2}$$



$$P=\int_{-\frac{1}{2}}^{0}\left[\left(\frac{1}{3}Y+\frac{2}{3}\right)-(-Y)\right]dY+\int_{0}^{1}\left[\left(\frac{1}{3}Y+\frac{2}{3}\right)-Y^3\right]dY=$$
$$=\int_{-\frac{1}{2}}^{0}\left(\frac{4}{3}Y+\frac{2}{3}\right)dY+\int_{0}^{1}\left(-Y^3+\frac{1}{3}Y+\frac{2}{3}\right)dY=$$
$$=\frac{4}{3}\cdot\frac{1}{2}Y^2\Big|_{-\frac{1}{2}}^{0}+\frac{2}{3}Y\Big|_{-\frac{1}{2}}^{0}-\frac{1}{4}Y^4\Big|_{0}^{1}+\frac{1}{3}\cdot\frac{1}{2}Y^2\Big|_{0}^{1}$$
$$+\frac{2}{3}Y\Big|_{0}^{1}=\frac{\cancel{2}}{3}\cdot\left(-\frac{1}{\cancel{4}_{2}}\right)+\frac{\cancel{2}}{3}\cdot\frac{1}{\cancel{2}}-\frac{1}{4}+\frac{1}{6}+$$
$$+\frac{2}{3}=\cancel{-\frac{1}{6}}+\frac{1}{3}-\frac{1}{4}\cancel{+\frac{1}{6}}+\frac{2}{3}=\frac{3}{4}$$

#) Izračunati površinu figure koja je određena linijama $Y=-2$, $Y=x^3+x$, $x+Y=3$.

Rj. $Y=-2$, $x+Y=3$ su prave linije i njih nije teško nacrtati. Problem za crtanje predstavlja kriva $Y=x^3+x$.

Ispitajmo f-ju $Y=x^3+x$. $\mathcal{D}: x\in\mathbb{R}$

$f(-x)=-x^3-x=-(x^3+x)$ f-ja je neparna

A(0,0) je nula f-je i presjek sa Y-osom

f-ja nema prekida ⇒ f-ja nema vertikalnu asimptotu

f-ja nema horizontalnu ni kosu asimptotu

$Y'=3x^2+1$ f-ja je uvijek pozitivna (raste za svako x)

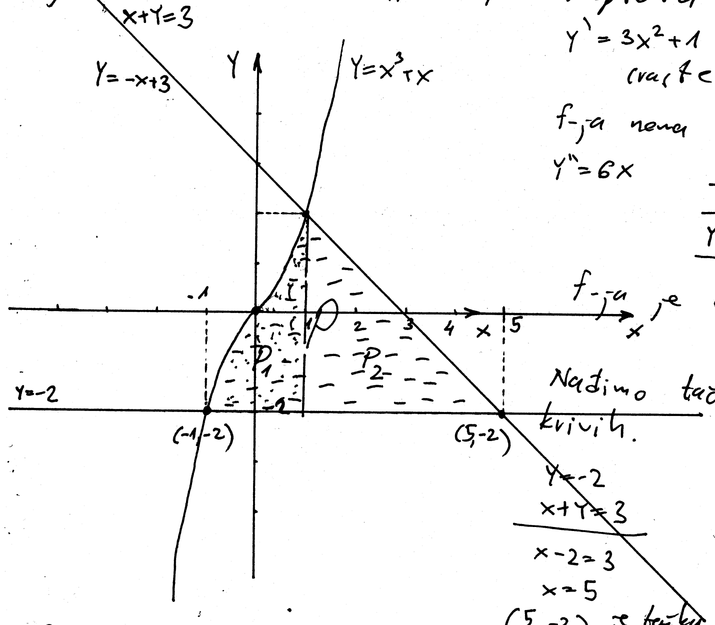
f-ja nema ekstrem

$Y''=6x$

x	(0,+∞)
Y''	+
Y	∪

(0,0) je prevojna tačka

f-ja je ovog oblika



Nađimo tačke presjeka datih krivih.

$$\frac{\begin{array}{l} Y=-2 \\ x+Y=3 \end{array}}{x-2=3}$$
$$x=5$$

(5,-2) je tačka presjeka

$$\frac{\begin{array}{l} Y=-2 \\ Y=x^3+x \end{array}}{-2=x^3+x}$$
$$x^3+x+2=0$$
$$x=-1:\ -1-1+2=0$$

$$(x^3+x+2):(x+1)=x^2-x+2$$
$$\underline{-\ x^3+x^2}$$
$$-x^2+x+2$$
$$\underline{-\ -x^2-x}$$
$$2x+2$$
$$\underline{2x+2}$$
$$//\ //$$

$$x^3+x+2=(x+1)\underbrace{(x^2-x+2)}_{>0\ \forall x}$$

Rješenje jednačine $x^3+x+2=0$ je $x=-1$.

(-1,-2) je tačka presjeka datih krivih

$Y = x^3 + x$

$x + Y = 3$

$Y = x^3 + x$

$Y = -x + 3$

$-x + 3 = x^3 + x$

$x^3 + 2x - 3 = 0$

$x = 1: \quad 1^3 + 2\cdot 1 - 3 = 3 - 3 = 0$

$(x^3 + 2x - 3) : (x - 1) = x^2 + x + 3$

$x^3 - x^2$

$x^2 + 2x - 3$

$-\; x^2 - x$

$3x - 3$

$-\; 3x - 3$

$= \quad =$

$x^3 + 2x - 3 = \underbrace{(x^2 + x + 3)}_{> 0\ \forall x}(x - 1)$

$(1, 2)$ je presječna tačka krivih

$$P_1 = \int_{-1}^{1} \left[(x^3 + x) - (-2)\right] dx = \int_{-1}^{1} (x^3 + x + 2)\, dx = \frac{1}{4}x^4\Big|_{-1}^{1} + \frac{1}{2}x^2\Big|_{-1}^{1} + 2x\Big|_{-1}^{1} = 4$$

$$P_2 = \int_{1}^{5} \left[(-x + 3) - (-2)\right] dx = \int_{1}^{5} (-x + 5)\, dx = -\frac{x^2}{2}\Big|_{1}^{5} + 5x\Big|_{1}^{5} =$$

$$= -\frac{1}{2}(25 - 1) + 5\cdot 4 = -\frac{1}{2}\cdot 24 + 20 = 20 - 12 = 8$$

$P = P_1 + P_2 = 8 + 4 = 12$ površina figure

(#) Izračunati površinu figure koju čine linije

$Y = (x-1)^2, \quad \frac{x^2}{1} - \frac{Y^2}{2} = 1.$

Rj. Da bi odredili granice za računanje površine potrebno je grafički predstaviti ove dvije linije.

ispitajmo f-ju $Y = (x-1)^2$

$D: x \in \mathbb{R}$

f-ja nije ni parna ni neparna

$f(0) = 1$, $(0, 1)$ je presjek sa Y-osom

$(x-1)^2 = 0 \Rightarrow x = 1$ $\quad (1, 0)$ je nula f-je

$Y = (x-1)^2 = x^2 - 2x + 1 \Rightarrow$ f-ja je oblika ∪

Nađimo još tjeme f-je

$Y' = 2x - 2$

$Y' = 0 \Rightarrow 2x - 2 = 0 \Rightarrow x = 1$

$T(1, 0)$

Prije nego grafički predstavimo liniju $Y = (x-1)^2$ pronađimo u kojim tačkama siječe liniju $\frac{x^2}{1} - \frac{Y^2}{2} = 1$.

$Y = x^2 - 2x + 1 = (x-1)^2$

$2x^2 - Y^2 = 2$

$Y^2 = (x^2 - 2x + 1)^2 =$

$= (x^2 - 2x + 1)(x^2 - 2x + 1) =$

$= \underline{x^4} \underline{\underline{- 2x^3}} + x^2 \underline{\underline{- 2x^3}} + 4x^2 - 2x + x^2 - 2x + 1$

$= x^4 - 4x^3 + 6x^2 - 4x + 1.$

$2x^2 - Y^2 = 2$

$\underline{2x^2} - x^4 + 4x^3 \underline{- 6x^2} + 4x - 1 - 2 = 0 \quad /\cdot(-1)$

$x^4 - 4x^3 + 4x^2 - 4x + 3 = 0$

Označimo ovo sa $g(x) = x^4 - 4x^3 + 4x^2 - 4x + 3$

Kako je $g(1) = 0 \Rightarrow g(x)$ je djeljivo sa $(x-1)$

$(x^4 - 4x^3 + 4x^2 - 4x + 3) : (x - 1) = x^3 - 3x^2 + x - 3$

$x^4 - x^3$

$-3x^3 + 4x^2 - 4x + 3$

$-\; -3x^3 + 3x^2$

$x^2 - 4x + 3$

$-\; x^2 - x$

$-3x + 3$

$-3x + 3$

$= \quad =$

$g(x) = \underbrace{(x^3 - 3x^2 + x - 3)}_{g_1(x)}(x - 1)$

$g_1(0) = -3$

$g_1(1) = 1 - 3 + 1 - 3 = -4$

$g_1(2) = 8 - 12 + 2 - 3 = -5$

$g_1(3) = 27 - 27 + 3 - 3$

$g_1(-2) = -8 - 12 - 2 - 3 = -25$

$g_1(-1) = -1 - 3 - 1 - 3 = -8$

$g_1(-3) = -27 - 27 - 3 - 3 < 0$

$\Rightarrow g_1(x)$ je djeljivo sa $x - 3$

$(x^3 - 3x^2 + x - 3) : (x - 3) = x^2 + 1$

$-\; x^3 - 3x^2$

$x - 3$

$-\; x - 3$

$= \quad =$

Prema tome $g(x) = (x^2 + 1)(x - 3)(x - 1)$

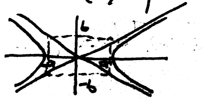
Za $x = 3 \Rightarrow Y = 4$

Za $x = 1 \Rightarrow Y = 0$

Presječne tačke krivih su $(3, 4)$ i $(1, 0)$

Krive oblika

$\frac{x^2}{a^2} - \frac{Y^2}{b^2} = 1$ zovemo HIPERBOLE i one su oblika

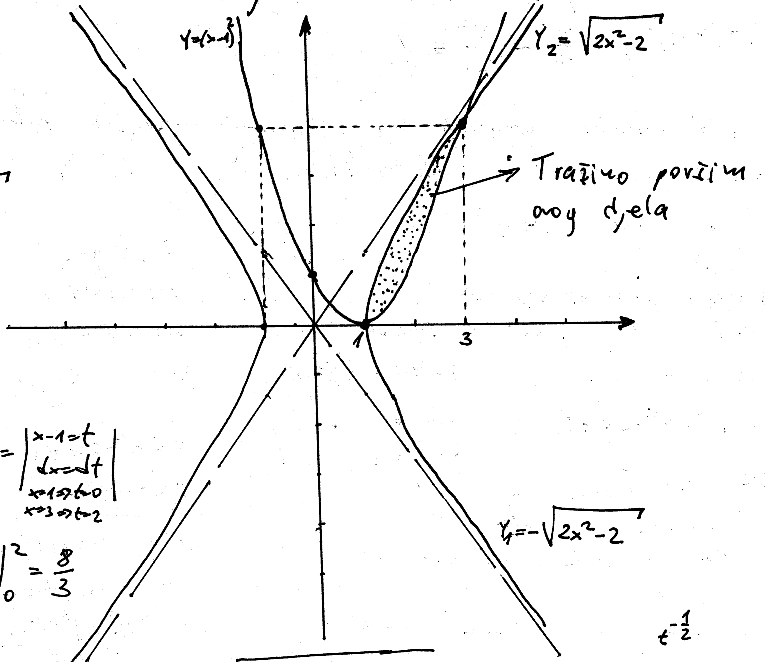


Nacrtajmo naše krive linije

$\sqrt{2} \approx 1{,}41$

$y^2 = 2x^2-2$

$y_{1,2} = \pm\sqrt{2x^2-2}$



$t^{-\frac{1}{2}}$

$$P_2 = \int_1^3 (x-1)^2\,dx = \left| \begin{matrix} x-1=t \\ dx=dt \\ x=1 \Rightarrow t=0 \\ x=3 \Rightarrow t=2 \end{matrix} \right| = \int_0^2 t^2\,dt = \frac{1}{3}t^3\Big|_0^2 = \frac{8}{3}$$

$$\int \frac{x}{\sqrt{x^2-1}}\,dx = \left| \begin{matrix} x^2-1=t \\ 2x\,dx=dt \\ x\,dx=\frac{1}{2}dt \end{matrix} \right| = \frac{1}{2}\int \frac{dt}{\sqrt{t}} = \frac{1}{2}\cdot\frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = \sqrt{t} + C = \sqrt{x^2-1} + C$$

$$P = \int_1^3 \Big(\underbrace{\sqrt{2x^2-2}}_{P_1} - \underbrace{(x-1)^2}_{P_2} \Big)\,dx$$

$$P_1 = \int_1^3 \sqrt{2}\cdot\sqrt{x^2-1}\,dx = \sqrt{2}\int_1^3 \frac{x^2-1}{\sqrt{x^2-1}}\,dx = \sqrt{2}\left(\int_1^3 \frac{x^2}{\sqrt{x^2-1}}\,dx - \int_1^3 \frac{dx}{\sqrt{x^2-1}} \right)$$

$$\int_1^3 x\cdot\frac{x}{\sqrt{x^2-1}}\,dx = \left| \begin{matrix} u=x & dv=\frac{x}{\sqrt{x^2-1}}\,dx \\ du=dx & v=\sqrt{x^2-1} \end{matrix} \right| = \underbrace{x\sqrt{x^2-1}\Big|_1^3}_{3\sqrt{8}-0} - \int_1^3 \sqrt{x^2-1}\,dx$$

$$\int_1^3 \frac{dx}{\sqrt{x^2-1}} = \ln|x+\sqrt{x^2-1}|\Big|_1^3 = \ln|3+\sqrt{8}|$$

$$\sqrt{2}\int_1^3 \sqrt{x^2-1}\,dx = \sqrt{2}\cdot\underbrace{3\sqrt{8}}_{6\sqrt{2}} - \sqrt{2}\int_1^3 \sqrt{x^2-1}\,dx - \sqrt{2}\ln(3+2\sqrt{2})$$

$$\int_1^3 \sqrt{2x^2-2}\,dx = 6 - \frac{\sqrt{2}\ln(2\sqrt{2}+3)}{2}$$

$$P = P_1 - P_2 = \frac{10}{3} - \frac{\sqrt{2}\ln(2\sqrt{2}+3)}{2}$$

tražena površina

Zadaci za vježbu

Površina ravne figure

2455. Izračunati površinu ograničenu linijama $y^2=2x+1$ i $x-y-1=0$.

2456. Izračunati površinu ograničenu parabolom $y=-x^2+4x-3$ i njenim tangentama u tačkama $(0;\ 3)$ i $(3;\ 0)$.

2457. Izračunati površinu ograničenu parabolom $y^2=2px$ i njenom normalom koja sa x-osom zaklapa ugao od 135°.

2458. Izračunati površinu ograničenu parabolama $y=x^2$ i $y=\sqrt{x}$.

2459. Izračunati površinu ograničenu parabolama $y^2+8x+16$ i $y^2-24x=48$.

2460. Izračunati površinu ograničenu parabolama $y=x^2$ i $y=\frac{x^3}{3}$.

2461. Parabola $y=\frac{x^2}{2}$ deli krug $x^2+y^2=8$ na dva dela; naći površinu svakog od njih.

2462. Parabola $y^2=6x$ deli krug $x^2+y^2=16$ na dva dela; naći njihove površine.

2463. Iz kruga poluprečnika a isečena je elipsa čija je velika osa jednaka jednom od prečnika kruga, a manja osa je $2b$. Dokazati da je površina preostalog dela kruga jednaka površini elipse čije su poluose a i $a-b$.

2464. Naći površinu ograničenu lukom hiperbole i njenom tetivom koja prolazi kroz žižu i normalna je na realnu osu.

2465. Hiperbola $x^2-2y^2=\frac{a^2}{4}$ deli krug $x^2+y^2=a^2$ na tri dela; naći površinu svakog od njih.

2466. Hiperbola $\frac{x^2}{2}-y^2+1$ deli oblast ograničenu elipsom $\frac{x^2}{4}+y^2=1$ na tri dela: izračunati njihove površine.

2467. Izračunati površinu ograničenu krivama $y=\frac{1}{1+x^2}$ i $y=\frac{x^2}{2}$.

2468. Izračunati površinu ograničenu krivom $y=x(x-1)^2$ i apscisnom osom.

2469. Naći površinu ograničenu krivom $x=y^2(y-1)$ i ordinatnom osom.

2470. Krive zadate jednačinama $y^m=x^n$ i $y^n=x^m$ u kojima su m i n celi pozitivni brojevi, ograničavaju jednu ili više (zatvorenih) oblasti. Izračunati površinu one od njih koja leži u prvom kvadrantu, a zatim i ukupnu površinu ograničenu ovim linijama vodeći računa o parnosti i neparnosti brojeva m i n.

2471. a) Izračunati površinu ograničenu krivom $y=x-x^2\sqrt{x}$ i x-osom. b) Izračunati površinu ograničenu dvema granama krive $(y-x)^2=x^5$ i pravom $x=4$.

2472. Izračunati površinu oblasti ograničene krivom $(y-x-2)^2=9x$ i koordinatnim osama.

2473. Izračunati površinu ograničenu petljom krive $y^2=x(x-1)^2$.

2474. Izračunati površinu ograničenu zatvorenom krivom $y^2=(1-x^2)^3$.

2475. Izračunati površinu ograničenu zatvorenom krivom $y^2=x^2-x^4$.

2476. Izračunati površinu ograničenu zatvorenom krivom $x^4-ax^3+a^2y^2=0$.

2477. Izračunati površinu konačne oblasti ograničene krivom $x^2y^2=4(x-1)$ i pravom koja prolazi kroz njene prevojne tačke.

2478. Izračunati površinu ograničenu krivama $y=e^x$, $y=e^{-x}$ i pravom $x=1$.

2479. Izračunati površinu konačne oblasti ograničene krivom $y=(x^2+2x)e^{-x}$ i x-osom.

2480. Izračunati površinu ograničenu krivom $y=e^{-x}(x^2+3x+1)+e^2$, x-osom i dvema pravama, paralelnim y-osi, koje prolaze kroz tačke ekstremuma funkcije y.

Rješenja

2455. $\frac{16}{3}$. **2456.** $\frac{9}{4}$. **2457.** $\frac{16}{3}p^2$.

2458. $\frac{1}{3}$. **2459.** $\frac{32}{3}\sqrt{6}$.

2460. $2\frac{1}{4}$. **2461.** $2\pi+\frac{4}{3}$ i $6\pi-\frac{4}{3}$.

2462. $\frac{4}{3}(4\pi+\sqrt{3})$ i $\frac{4}{3}(8\pi-\sqrt{3})$.

2464. $\frac{b^2c}{a}-ab\ln\frac{c+b}{a}=b\,[\varepsilon b-a\ln(\varepsilon+\sqrt{\varepsilon^2-1})]$, gde je ε ekscentricitet.

2465. $a^2\left[\frac{\pi}{6}-\frac{\sqrt{2}}{8}\ln(\sqrt{3}+\sqrt{2})\right]$;

$a^2\left[\frac{\pi}{6}-\frac{\sqrt{2}}{8}\ln(\sqrt{3}+\sqrt{2})\right]$

i $a^2\left[\frac{2\pi}{3}+\frac{\sqrt{2}}{4}\ln(\sqrt{3}+\sqrt{2})\right]$.

2466. $S_1=S_3=\pi-\frac{\sqrt{2}}{2}\ln 3-2\arcsin\sqrt{\frac{2}{3}}\approx 0{,}46$.

$S_2=2(\pi-S_1)$, **2467.** $\frac{\pi}{2}-\frac{1}{3}$.

2468. $\frac{1}{12}$. **2469.** $\frac{1}{12}$.

2470. $\left|\frac{m-n}{m+n}\right|$; $4\left|\frac{m-n}{m+n}\right|$, ako su brojevi m i n oba parni; $2\left|\frac{m-n}{m+n}\right|$, ako su m i n —oba neparni; $\left|\frac{m-n}{m+n}\right|$, ako je jedan od brojeva m i n paran a drugi neparan.

2471. a) $\frac{3}{14}$; b) $73\frac{1}{7}$.

2472. 1. (oblast se sastoji iz dva dela čije su površine jednake među sobom.

2473. $\frac{8}{15}$. **2474.** $\frac{3}{4}\pi$.

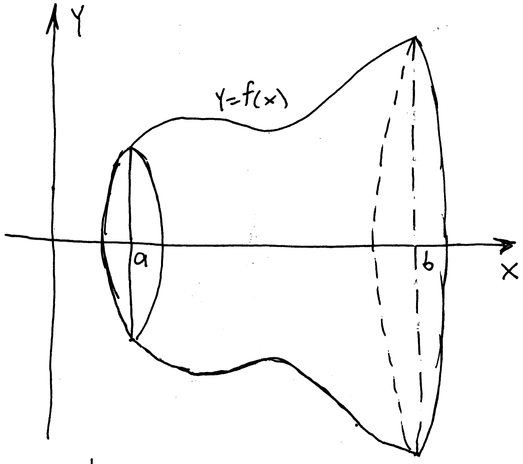
2475. $\frac{4}{}$. **2476.** $\frac{\pi a^2}{8}$.

2477. $8\left(\sqrt{1+\frac{2}{3}\sqrt{3}}-\operatorname{arctg}\sqrt{1+\frac{2}{3}\sqrt{3}}\right)$.

2478. $e+\frac{1}{e}-2$.

2479. 4. **2480.** $\frac{3}{e}(e^3-4)$.

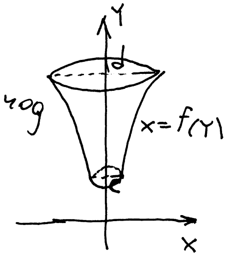
II Zapremina rotacionog tijela



zapremina tijela dobijenog rotacijom dijela krive $Y=f(x)$ oko x-ose

$$V_x=\pi\cdot\int_a^b [f(x)]^2\,dx$$

$$V_y=\pi\int_c^d [f(y)]^2\,dy$$ -zapremina tijela dobijenog rotacijom dijela krive $x=f(y)$ oko y-ose



Ako je kriva data u parametarskom obliku:

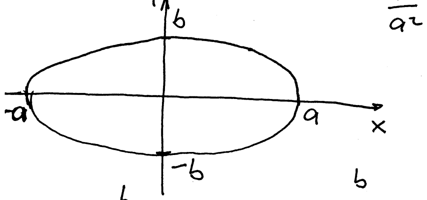
$x=\alpha(t)$
$Y=\beta(t)$
$t_1\leq t\leq t_2$

$$V_x=\pi\int_{t_1}^{t_2}[\beta(t)]^2\,|\alpha'(t)|\,dt$$

$$V_y=\pi\int_{t_1}^{t_2}[\alpha(t)]^2\,|\beta'(t)|\,dt$$

① Izračunati zapreminu tijela koje nastaje rotacijom krive $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ oko Y-ose.

Rj.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$x^2=a^2\cdot\left(1-\frac{y^2}{b^2}\right)$$

$$x^2=a^2\cdot\frac{b^2-y^2}{b^2}$$

$$x^2=\frac{a^2}{b^2}\cdot(b^2-y^2)$$

$$x=\pm\frac{a}{b}\sqrt{b^2-y^2}$$

$$V_Y=\pi\int_{-b}^{b}[f(y)]^2\,dy=\pi\int_{-b}^{b}\frac{a^2}{b^2}(b^2-y^2)\,dy=2\pi\frac{a^2}{b^2}\int_0^b(b^2-y^2)\,dy=$$

↖ Parna f-ja (simetrična u odnosu na Y-osu)

$$=2\pi\frac{a^2}{b^2}\left(b^2y\Big|_0^b-\frac{y^3}{3}\Big|_0^b\right)=2\pi\frac{a^2}{b^2}\left(b^3-\frac{1}{3}b^3\right)=2\pi\frac{a^2}{b^2}\cdot\frac{2}{3}b^3=\frac{4\pi a^2b}{3}.$$

② Figura u ravni ograničena parabolom $Y=4-x^2$ i pravama $Y\geqslant 3x$, $Y\geqslant 0$ rotira oko x-ose. Izračunati zapreminu dobijenog tijela.

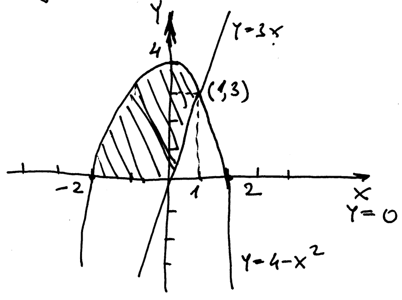
Rj.

$Y=4-x^2$
$Y=3x$

$3x=4-x^2$
$x^2+3x-4=0$
$D=9+16=25$
$x_{1,2}=\frac{-3\pm5}{2}$

$x_1=-4\Rightarrow Y_1=-12$
$x_2=1\Rightarrow Y_2=3$

$A(-4,-12)$ i $B(1,3)$ su tačke presjeka prave i parabole



$$V_x=V_1-V_2,\quad V_1=\pi\int_{-2}^{1}(4-x^2)^2\,dx,\quad V_2=\pi\int_0^1(3x)^2\,dx$$

$$V_1=\pi\int_{-2}^{1}(16-8x^2+x^4)\,dx=\pi\left(16x\Big|_{-2}^{1}-8\frac{x^3}{3}\Big|_{-2}^{1}+\frac{x^5}{5}\Big|_{-2}^{1}\right)=\pi\left(16\cdot3-\frac{8}{3}\cdot9+\frac{1}{5}\cdot33\right)$$

$1-(-2)$ $\quad 1^3-(-2)^3$ $\quad 1^5-(-2)^5$

$$=\pi\left(48-24+\frac{33}{5}\right)=\frac{158}{5}\pi$$

$$V_2=\pi\cdot9\int_0^1x^2\,dx=9\pi\frac{x^3}{3}\Big|_0^1=3\pi(1-0)=3\pi$$

$$V=V_1-V_2=\frac{158}{5}\pi-3\pi=\frac{158\pi-15\pi}{5}=\frac{138}{5}\pi$$

③ Izračunati zapreminu tijela nastalog obrtanjem oko x-ose figure omeđenu krivom $Y=\arcsin x$ i pravama $x=1$ i $Y=0$. Uputa: parcijalna integracija 2x

④ Izračunati zapreminu tijela koje nastaje rotacijom ravne figure ograničene parabolom $Y=6-x-x^2$ i prave $Y=0$ oko x-ose.

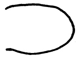
Ⓗ Izračunati zapreminu tijela koje nastaje rotacijom figure određene parabolom $y^2=9-3x$, tangentom na tu parabolu u tački $A(0,3)$ i x-osom oko x-ose.

Rj.

$y^2=9-3x$

$3x=-y^2+9$

$x=-\frac{1}{3}y^2+3$



f-ja je ovog oblika

$x'=-\frac{2}{3}y$

$x'=0$ akko $y=0$

$T(3,0)$ je tjeme f-je

$\cancel{x}=0 \Rightarrow y=\pm 3$

Koeficijent pravca tangente

$k=x'(A)=-\frac{2}{3}\cdot 3=-2$

$x-x_1=k(y-y_1)$

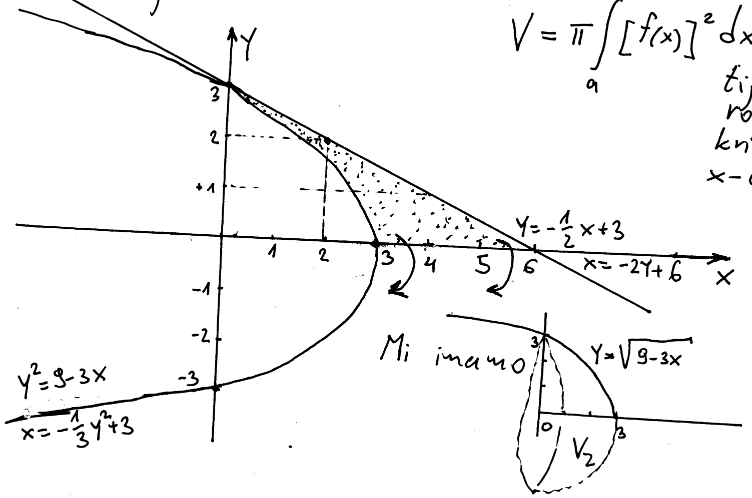
$x-0=(-2)(y-3)$

$x=-2y+6$

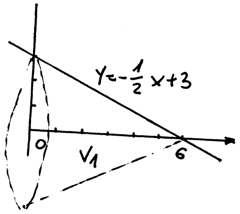
$-2y=x-6$

$y=-\frac{1}{2}x+3$

$$V=\pi\int_a^b[f(x)]^2dx$$ je zapremina tijela dobijena rotacijom dijela krive $y=f(x)$ oko x-ose



Mi imamo



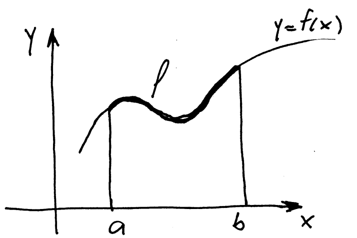
Zapremina našeg tijela se računa po formuli

$$V=V_1-V_2=\pi\int_0^6\left(-\frac{1}{2}x+3\right)^2dx-\pi\int_0^3\left(\sqrt{9-3x}\right)^2dx \overset{(1)\,i\,(2)}{=\!=} \frac{9}{2}\pi$$

$$V_1=\pi\int_0^6\left(\frac{1}{4}x^2-3x+9\right)dx=\pi\left(\frac{1}{4}\cdot\frac{1}{3}x^3\Big|_0^6-3\cdot\frac{1}{2}x^2\Big|_0^6+9x\Big|_0^6\right)=\pi(18-54+54)=18\pi \quad \ldots(1)$$

$$V_2=\pi\int_0^3(9-3x)dx=\pi\left(9x\Big|_0^3-3\cdot\frac{1}{2}x^2\Big|_0^3\right)=\pi\left(27-\frac{27}{2}\right)=\frac{27}{2}\pi \quad \ldots(2)$$

III Dužina luka krive



$$\ell=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$$

dužina luka krive $y=f(x)$

$$\ell=\int_c^d\sqrt{1+[f'(y)]^2}\,dy$$

dužina luka krive $x=f(y)$

Ako je kriva data u parametarskom obliku:

$x=x(t)$
$y=y(t)$
$t_1\le t\le t_2$

$$\Rightarrow \quad \ell=\int_{t_1}^{t_2}\sqrt{(\dot{x})^2+(\dot{y})^2}\,dt$$

gdje je $\dot{x}=\frac{dx}{dt}$

i $\dot{y}=\frac{dy}{dt}$ (izvod po t)

① Izračunati dužinu luka krive $y=\frac{x^2}{2}-\frac{\ln x}{4}$ ako je $1\le x\le 3$.

Rj. $\ell=\int_a^b\sqrt{1+[f'(x)]^2}\,dx$, $\quad y=f(x)=\frac{x^2}{2}-\frac{\ln x}{4}=\frac{1}{2}x^2-\frac{1}{4}\ln x$

$$y'=\frac{1}{2}\cdot 2x-\frac{1}{4}\cdot\frac{1}{x}=x-\frac{1}{4x}$$

$$\ell=\int_1^3\sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,dx=\int_1^3\sqrt{1+x^2-\cancel{2x}\cdot\frac{1}{\underset{2}{\cancel{4x}}}+\frac{1}{16x^2}}\,dx=\int_1^3\sqrt{x^2+\frac{1}{2}+\frac{1}{16x^2}}$$

$$=\int_1^3\sqrt{\left(x+\frac{1}{4x}\right)^2}\,dx=\int_1^3\left(x+\frac{1}{4x}\right)dx=\frac{x^2}{2}\Big|_1^3+\frac{1}{4}\ln x\Big|_1^3=\frac{1}{2}(9-1)+\frac{1}{4}(\ln 3-\ln 1)$$

$$=4+\frac{1}{4}\ln 3=4+\ln\sqrt[4]{3}.$$

$$\boxed{\frac{1}{4}\ln 3=\ln 3^{\frac{1}{4}}}$$

② Naći dužinu luka kojeg na paraboli $y^2=2x+1$ odsjeca prava $x-y=1$.

Rj.

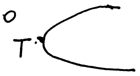
$$y^2=2x+1$$
$$x-y=1$$
$$\overline{\qquad}$$
$$y^2=2x+1$$
$$y=x-1$$
$$\overline{\qquad}$$
$$(x-1)^2=2x+1$$

$$x^2-2x+1=2x+1$$
$$x^2-4x=0$$
$$x(x-4)=0$$
$$x_1=0 \Rightarrow y_1=-1$$
$$x_2=4 \Rightarrow y_2=3$$

$A(0,-1)$ i $B(4,3)$ su tačke presjeka parabole i prave

$$y^2=2x+1 \qquad x=0 \Rightarrow y=\pm1$$
$$2x=y^2-1$$
$$x=\frac{1}{2}y^2-\frac{1}{2}$$

$a>0$

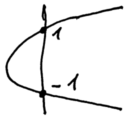


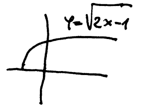
$T(-\frac{1}{2},0)$

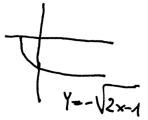
$D=1$

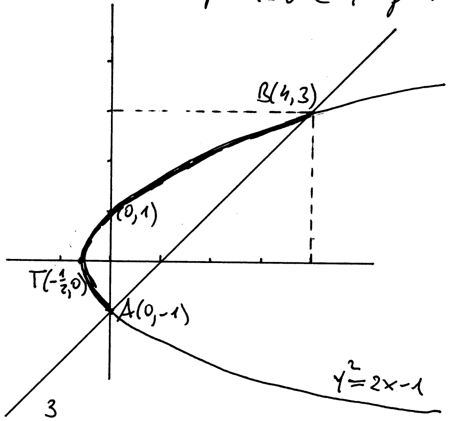
$$-\frac{D}{4a}=-\frac{1}{4\cdot\frac{1}{2}}=-\frac{1}{2}, \quad -\frac{b}{2a}=0$$

$$y^2=2x+1$$
$$y=\pm\sqrt{2x-1}$$









$$y^2=2x+1$$
$$x=\frac{1}{2}y^2-\frac{1}{2}$$
$$x'=\frac{1}{2}\cdot2y=y \quad tj. \quad x'_y=y$$

$$\ell=\int_{-1}^{3}\sqrt{1+y^2}\,dy$$

integral $\int\sqrt{1+y^2}\,dy$ smo uradili Metodom Ostrogradskog, 3 zadatak na 69 strani u skripti (umjesto y imali smo x)

$$\ell=\int_{-1}^{3}\sqrt{1+y^2}\,dy=\frac{1}{2}y\sqrt{y^2+1}\Big|_{-1}^{3}+\frac{1}{2}\ln|y+\sqrt{y^2+1}|\Big|_{-1}^{3}=$$

$$=\frac{1}{2}\left(3\sqrt{10}-(-1)\sqrt{2}\right)+\frac{1}{2}\left(\ln|3+\sqrt{10}|-\ln|-1+\sqrt{2}|\right)=$$

$$=\frac{3\sqrt{10}+\sqrt{2}}{2}+\frac{1}{2}\ln\left|\frac{3+\sqrt{10}}{\sqrt{2}-1}\cdot\frac{\sqrt{2}+1}{\sqrt{2}+1}\right|=\frac{3\sqrt{10}+\sqrt{2}}{2}+\frac{1}{2}\ln|(3+\sqrt{10})(\sqrt{2}+1)|$$

$$=\frac{3\sqrt{10}-\sqrt{2}}{2}+\ln\sqrt{(3+\sqrt{10})(\sqrt{2}+1)}$$

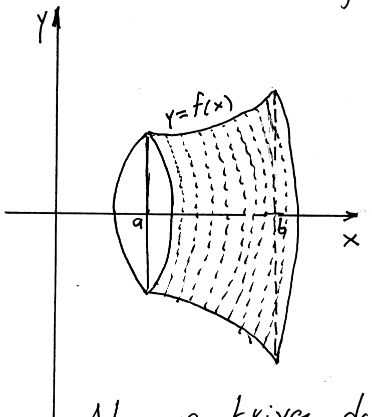
③ Izračunati dužinu luka krive

a) $y=\sqrt{2x-x^2}-1$, ako je $\frac{1}{4}\le x\le1$

b) $x=\frac{1}{4}y^2-\frac{1}{2}\ln y$, ako je $1\le y\le e$

IV Komplanacija obrtne površi

komplanacija lat. postupak za izračunavanje površina dijelova zakrivljenih ploha



površina omotača tijela dobijenog rotacijom dijela krive $y=f(x)$ oko x-ose

$$P=2\pi\int_a^b|f(x)|\cdot\sqrt{1+(f'(x))^2}\,dx$$

Ako je kriva data u parametarskom obliku

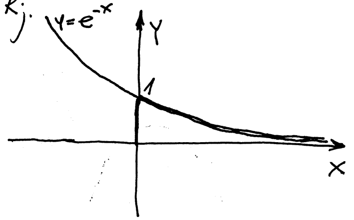
$x=\alpha(t)$
$y=\beta(t)$
$t_1\le t\le t_2$

$$\Rightarrow \quad P=2\pi\int_{t_1}^{t_2}|\beta(t)|\cdot\sqrt{(\dot{x})^2+(\dot{y})^2}\,dt$$

gdje je $\dot{x}=\frac{dx}{dt}$, $\dot{y}=\frac{dy}{dt}$

① Izračunati površinu omotača tijela koje nastaje rotacijom krive $y=e^{-x}$ oko x-ose za $x\ge0$.

Rj.



$$y=e^{-x}, \quad y'=e^{-x}\cdot(-1)=-e^{-x}$$

$$P=2\pi\int_0^{+\infty}e^{-x}\cdot\sqrt{1+e^{-2x}}\,dx$$

$$=2\pi\lim_{R\to+\infty}\int_0^{R}e^{-x}\sqrt{1+e^{-2x}}\,dx$$

$$\int_0^R e^{-x}\sqrt{1+e^{-2x}}\,dx = \left|\begin{array}{ll} u=e^{-x} & x=0 \Rightarrow u=1 \\ du=-e^{-x}dx & x=R \Rightarrow u=e^{-R} \\ -du=e^{-x}dx & \end{array}\right| = \int_1^{e^{-R}}\sqrt{1+u^2}\cdot(-du)$$

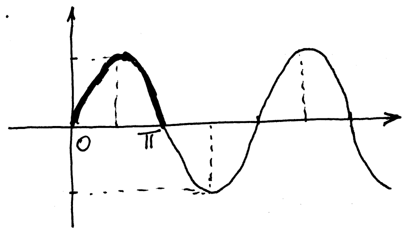
$$= -\int_1^{e^{-R}}\sqrt{1+u^2}\,du = \int_{e^{-R}}^1\sqrt{1+u^2}\,du \underset{\text{(metoda Ostrograd.)}}{\overset{\text{rađeno ranije}}{=}} \frac{1}{2}u\sqrt{1+u^2}\Big|_{e^{-R}}^1 + \frac{1}{2}\ln\left|x+\sqrt{1+x^2}\right|\Big|_{e^{-R}}^1$$

$$= \frac{\sqrt2}{2} - \frac12 e^{-R}\sqrt{1+e^{-2R}} + \frac12\ln(1+\sqrt2) - \frac12\ln\left(e^{-R}+\sqrt{1+e^{-2R}}\right), \qquad \begin{array}{l} e^{-2R}\to 0,\ R\to\infty \\ e^{-R}\to 0,\ R\to\infty \end{array}$$

$$P = 2\pi\lim_{R\to+\infty}\int_0^R e^{-x}\sqrt{1+e^{-2x}}\,dx = 2\pi\left(\frac12\sqrt2+\frac12\ln(1+\sqrt2)\right) = \pi\left(\sqrt2+\ln(1+\sqrt2)\right)$$

② Izračunati površinu omotača tijela koje nastaje rotacijom jednog svoda sinusoide $y=\sin x$ oko x-ose.

Kj.



$y=\sin x$

$y'=\cos x$

$$P = 2\pi\int_0^\pi \sin x\cdot\sqrt{1+\cos^2 x}\,dx =$$

$$= \left|\begin{array}{ll} \cos x = t & x=0 \Rightarrow t=1 \\ -\sin x\,dx = dt & x=\pi \Rightarrow t=-1 \\ \sin x\,dx = -dt & \end{array}\right| = 2\pi\int_1^{-1}\sqrt{1+t^2}\,(-dt) = -2\pi\int_1^{-1}\sqrt{1+t^2}\,dt$$

$$= 2\pi\int_{-1}^1 \underset{\searrow \text{parna f-ja}}{\sqrt{1+t^2}}\,dt = 4\pi\int_0^1\sqrt{1+t^2}\,dt \underset{\text{(metoda Ostrogradskog)}}{\overset{\text{rađeno ranije}}{=}} 4\pi\left[\frac12 t\sqrt{1+t^2}\Big|_0^1 + \frac12\ln\left|t+\sqrt{1+t^2}\right|\Big|_0^1\right]$$

$$= 4\pi\cdot\frac{\sqrt2}{2} + 4\pi\cdot\frac12\left(\ln(1+\sqrt2) - \underbrace{\ln 1}_{0}\right) = 2\sqrt2\pi + 2\pi\ln(1+\sqrt2)$$

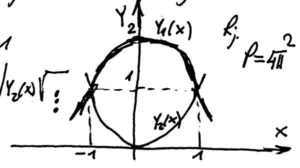
③ Izračunati površinu tijela koje nastaje rotacijom kružnice $x^2+(y-1)^2=1$ oko x-ose.

Uputa: $Y_1(x) = 1+\sqrt{1-x^2}$

$Y_2(x) = 1-\sqrt{1-x^2}$

$$P = P_1+P_2 = 2\pi\int_{-1}^1 Y_1(x)\sqrt{1+(Y_1')^2}\,dx + \int_{-1}^1 Y_2(x)\sqrt{\ }\ \vdots$$

tijelo koje nastaje rotacijom TORUS (ŠLAUF)



Kj. $P=4\pi^2$

Zadaci za vježbu

Dužina luka krive*).

2519. Izračunati dužinu luka lančanice $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ (od $x_1=0$ do $x_2=b$).

2520. Izračunati dužinu luka parabole $y^2=2px$ od temena do tačke $M(x,y)$. (Za nezavisno promenljivu uzeti y.)

2521. Naći dužinu luka krive $y=\ln x$ (od $x_1=\sqrt3$ do $x_2=\sqrt8$).

2522. Naći dužinu luka krive $y=\ln(1-x^2)$ $\left(\text{od } x_1=0 \text{ do } x_2=\frac12\right)$.

2523. Naći dužinu luka krive $y=\ln\frac{e^x+1}{e^x-1}$ (od $x_1=a$ do $x_2=b$).

2524. Izračunati dužinu onog dela semikubne parabole $y^2=\frac23(x-1)^3$, koji leži unutar parabole $y^2=\frac{x}{3}$.

2525. Izračunati dužinu onog dela semikubne parabole $5y^3=x^2$, koji leži unutar kruga $x^2+y^2=6$.

2526. Izračunati dužinu petlje krive $9ay^2=x(x-3a)^2$.

2527. Naći obim jednog od krivolinijskih trouglova koji obrazuju apscisna osa i krive $y=\ln\cos x$ i $y=\ln\sin x$.

2528. Naći dužinu luka krive $y=\frac{x^2}{4}-\frac{\ln x}{2}$, između njene najniže tačke i temena (teme je tačka na krivoj u kojoj krivina krive dostiže ekstremnu vrednost).

2529. Naći dužinu krive $y=\sqrt{x-x^2}+\arcsin\sqrt{x}$.

2530. Naći dužinu krive $(y-\arcsin x)^2=1-x^2$.

2531. Na cikloidi $x=a(t-\sin t$ [illegible] dužinu prvog svoda cikloide u odnos [illegible]

2532. Tačke $A(R, 0)$ i $B(0, R)$ [illegible] na astroidi $x=R\cos^3 t$, $y=R\sin^3 t$, odrediti na njoj tačku M koja deli dužinu luka $\widehat{AB}$ u odnosu $1:3$.

2533*. $4\frac{a^2+ab+b^2}{a+b}$. Staviti $x=a\cos^3 t$, $y=b\sin^3 t$.

2533*. Naći dužinu luka krive

$$\left(\frac{x}{a}\right)^{\frac23}+\left(\frac{y}{b}\right)^{\frac23}=1.$$

2534. Naći dužinu luka krive $x=a\cos^5 t$, $y=a\sin^5 t$.

2535. Naći dužinu luka traktrise $x=a\left(\cos t+\ln\operatorname{tg}\frac{t}{2}\right)$, $y=a\sin t$ od njene tačke $(0, a)$ do njene tačke (x, y).

2536. Naći dužinu luka evolvente kruga

$$x=R(\cos t+t\sin t),\quad y=R(\sin t-t\cos t).$$

(od $t_1=0$ do $t_2=\pi$).

2537. Izračunati dužinu luka krive

$$x=(t^2-2)\sin t+2t\cos t,\quad y=(2-t^2)\cos t+2t\sin t.$$

(od $t_1=0$ do $t_2=\pi$).

2538. Naći dužinu petlje krive $x=t^2$, $y=t-\frac{t^3}{3}$.

2539. Po krugu poluprečnika a, spolja i iznutra, kotrljaju se (bez klizanja) istim ugaonim brzinama dva kruga jednakih poluprečnika b. U trenutku $t=0$ oni svojim tačkama M_1 i M_2 dodiruju nepomični krug u tački M. Pokazati da dužina puteva koje pređu tačke M_1 i M_2 za proizvoljni interval vremena t, stoje u stalnom odnosu čija je vrednost $\frac{a+b}{a-b}$ (vidi zadatak 2493).

2540. Dokazati da dužina onog dela krive $x=f''(t)\cos t+f'(t)\sin t$, $y=-f''(t)\sin t+f'(t)\cos t$, koji odgovara intervalu (t_1, t_2), iznosi

$$[f(t)+f''(t)]\,\Big|_{t_1}^{t_2}.$$

Rješenja

2519. $\frac{a}{2}\left(e^{\frac{b}{a}}-e^{-\frac{b}{a}}\right)$.

2520. $\frac{y}{2p}\sqrt{y^2+p^2}+\frac{p}{2}\ln\frac{y+\sqrt{y^2+p^2}}{p}$.

2521. $1+\frac12\ln\frac32$.

2522. $\ln 3-\frac12$. **2523.** $\ln\frac{e^b-e^{-b}}{e^a-e^{-a}}$.

2524. $\frac89\left(\frac52\sqrt{\frac52}-1\right)$.

2525. $2\frac{26}{27}$. **2526.** $4a\sqrt3$.

2527. $\frac{\pi}{2}+2\ln\operatorname{tg}\frac{3\pi}{8}-\frac{\pi}{2}+2\ln(\sqrt2+1)$.

2528. $\frac16+\frac14\ln 3$. **2529.** 2. **2530.** 8.

2531. za $t=\frac{2\pi}{3}$; $\left[x=a\left(\frac{2\pi}{3}-\frac{\sqrt3}{2}\right),\quad y=\frac{3a}{2}\right]$.

2532. za $t=\frac{\pi}{6}$, $\left(x=\frac{3\sqrt3}{8}R,\ y=\frac{R}{8}\right)$.

2533*. $4\frac{a^2+ab+b^2}{a+b}$. Staviti $x=a\cos^3 t$, $y=b\sin^3 t$.

2534. $5a\left[1+\frac{1}{2\sqrt3}\ln(2+\sqrt3)\right]$.

2535. $a\ln\frac{a}{y}$. **2536.** $\frac{\pi^2}{2}R$.

2537. $\frac{\pi^3}{3}$. **2538.** $4\sqrt3$.

Zadaci za vježbu

2541. Primeniti rezultat prethodnog zadatka na izračunavanje dužine luka krive $x=e^t(\cos t-\sin t,\ y=e^t(\cos t-\sin t)$ od $t_1=0$ do $t_2=t$).

2542. Dokazati da luci krivih

$$x=f(t)-\varphi'(t),\quad y=\varphi(t)+f'(t)$$

i

$$x=f'(t)\sin t-\varphi'(t)\cos t,\quad y=f'(t)\cos t+\varphi'(t)\sin t$$

koji odgovaraju jednom istom intervalu parametra t, imaju jednake dužine.

2543. Naći dužinu luka Arhimedove spirale $\rho=a\varphi$ od početne do završne tačke prvog zavoja.

2544. Dokazati da luk parabole $y=\frac{1}{2p}x^2$ koji odgovara intervalu $0\leqslant x\leqslant a$, ima istu dužinu kao i luk spirale $\rho=p\varphi$ koji odgovara intervalu $0\leqslant\rho\leqslant a$.

2545. Izračunati dužinu luka hiperbolične spirale $\rho\varphi=1$ $\left(\text{od } \varphi_1=\frac{3}{4} \text{ do } \varphi_2=\frac{4}{3}\right)$.

2546. Naći dužinu kardioide $\rho=a(1+\cos\varphi)$.

2547. Naći dužinu krive $\rho=a\sin^3\frac{\varphi}{3}$ (vidi zad. 2505).

2548. Dokazati da je dužina krive $\rho=a\sin^m\frac{\varphi}{m}$ (m je ceo broj) samerljiva sa a ako je m paran broj, a samerljiva sa dužinom kruga poluprečnika a ako je m neparan broj.

2549. Za koje se vrednosti izložitelja k ($k\neq 0$) dužina luka krive $y=ax^k$ može izraziti pomoću elementarnih funkcija? (Pozvati se na Čebiševljevu teoremu o integrabilnosti (u konačnom vidu) diferencijalnog binoma).

2550. Naći dužinu krive zadate jednačinom $y=\int_{\pi}^{x}\sqrt{\cos x}\,dx$.

2551. Izračunati dužinu luka krive $L=4\int_0^{\frac{\pi}{4}}\left(\sqrt{a^2\cos^2 t+b^2\sin^2 t}+\sqrt{a^2\sin^2 t+b^2\cos^2 t}\right)dt$, ordinatnog početka do najbliže tačke u kojoj je tangenta krive paralelna ordinatnoj osi.

2552. Uveriti se da je dužina luka sinusoide $y=\sin x$ koji odgovara jednom njenom periodu, jednaka dužini elipse čije su poluose $\sqrt{2}$ i 1.

2553. Pokazati da je dužina luka „prikraćene" ili „izdužene" cikloide $x=mt-n\sin t$, $y=m-n\cos t$ (m i n su pozitivni brojevi) u intervalu od $t_1=0$ do $t_2=2\pi$, jednaka dužini elipse sa poluosama $a=m+n$ i $b=|m-n|$.

2554*. Dokazati da dužina elipse sa poluosama a i b zadovoljava nejednakosti $\pi(a+b)<L<\pi\sqrt{2}\cdot\sqrt{a^2+b^2}$ (zadatak Johana Bernulija).

Rješenja

2541. $2(e^t-1)$.

2543. $\pi a\sqrt{1+4\pi^2}+\frac{a}{2}\ln(2\pi+\sqrt{1+4\pi^2})$.

2545. $\ln\frac{3}{2}+\frac{5}{12}$.

2546. $8a$. **2547.** $\frac{3}{2}\pi a$.

549. k mora imati oblik $\frac{2N+1}{2N}$ ili $\frac{2N}{2N-1}$, pri čemu je N ceo broj.

2550. 4. **2551.** $\ln\frac{\pi}{2}$

2554*. Dokazati da se dužina elipse

može predstaviti u obliku

i primeniti teoremu o proceni integrala.

Zadaci za vježbu

Zapremina tela

2555. Izračunati zapreminu tela ograničenog površinom koja nastaje obrtanjem parabole $y^2=4x$ oko njene ose (obrtni paraboloid), i ravni normalnom na tu osu, postavljenom na odstojanju 1 od temena parabole.

2556. Elipsa čija je velika osa $2a$, a mala $2b$ obrće se: a) oko velike ose, b) oko male ose; naći zapremine dobijenih obrtnih elipsoida. Kao specijalan slučaj izvesti otuda obrazac za zapreminu lopte.

2557. Simetričan parabolični segment čija je osnovica a i visina h, obrće se oko osnovice; izračunati zapreminu tako nastalog obrtnog tela (Kavaljerijev „limun").

2558. Figura koju obrazuju hiperbola $x^2-y^2=a^2$ i prava $x=a+h$ ($h>0$) obrće se oko apscisne ose; naći zapreminu obrtnog tela.

2559. Figura koju obrazuju kriva $y=xe^x$, prava $x=1$ i x-osa obrće se oko x-ose; naći zapreminu tako nastalog obrtnog tela.

2560. Lančanica $y=\frac{e^x+e^{-x}}{2}$ se obrće oko apscisne ose i obrazuje površinu koja se naziva katenoid; naći zapreminu tela ograničenog katenoidom i dvema ravnima normalnim na apscisnu osu i udaljenim od koordinatnog početka za a i b jedinica.

2561. Figura koju obrazuju luci parabola $y=x^2$ i $y^2=x$ obrće se oko apscisne ose; izračunati zapreminu tako nastalog obrtnog tela.

2562. Naći zapreminu tela koje nastaje obrtanjem petlje krive $(x-4)y^2=$ $=x(x-3)$ oko apscisne ose.

2563. „Krivolinijski trapez", sa osnovicom [0, 1], ograničen lukom krive $y=\arcsin x$, obrće se oko x-ose; naći zapreminu tako nastalog tela.

2564. Segment parabole $y=2x-x^2$, čija je osnovica [0, 2], obrće se oko ordinatne ose; izračunati zapreminu tako nastalog tela.

2565. Figura koju obrazuju luk krive $y=\sin x$ i odsečak $[0, \pi]$ apscisne ose, obrće se oko ordinatne ose; naći zapreminu tako nastalog tela.

2566. Naći zapreminu tela koje nastaje obrtanjem leminskate $(x^2+y^2)^2=$ $=a^2(x^2-y^2)$ oko apscisne ose.

2567. Izračunati zapreminu tela koje nastaje obrtanjem krive:
1) $x^4+y^4=a^2x^2$; 2) $x^4+y^4=x^3$.

2568. Jedan svod cikloide $x=a(t-\sin t)$, $y=a(1-\cos t)$ obrće se oko svoje tetive; izračunati zapreminu tako nastalog tela.

2569. Svod cikloide (vidi prethodni zadatak) zajedno sa svojom tetivom obrće se oko svoje ose simetrije; naći zapreminu tako nastalog tela.

2570. Naći zapreminu tela koje nastaje obrtanjem astroide $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ oko svoje ose simetrije.

2571. Deo krive $x=\frac{c^2}{a}\cos^3 t$, $y=\frac{c^2}{b}\sin^3 t$ (evoluta elipse) koji leži u prvom kvadrantu, zajedno sa odgovarajućim odsečcima koordinatnih osa, obrće se oko apscisne ose; naći zapreminu tako nastalog tela.

2572. Izračunati zapreminu beskrajnog vretena koje nastaje obrtanjem krive $y=\frac{1}{1+x^2}$ oko svoje asimptote.

2573. Izračunati zapreminu tela koje nastaje obrtanjem krive $y^2=2exe^{-2x}$ oko svoje asimptote.

2574*. Figura koju obrazuju kriva $y=e^{-x^2}$ i njena asimptota obrće se jedanput oko apscisne, a drugi put oko ordinatne ose; izračunati zapremine tako dobijenih tela.

2575*. Izračunati zapreminu tela koje nastaje obrtanjem krive $y=x^2e^{-x^2}$ oko svoje asimptote.

Rješenja

2555. 2π. **2556.** 1) $\frac{4}{3}\pi ab^2$; 2) $\frac{4}{3}\pi a^2b$

2557. $\frac{8}{15}\pi h^2a$

2558. $\frac{\pi h^2}{3}(3a+h)$. **2559.** $\frac{\pi}{4}e^2-1)$.

2560. $\frac{\pi}{4}\left[\frac{e^{2b}-e^{-2b}}{2}-\frac{e^{2a}-e^{-2a}}{2}+(b-a)\right]$.

2561. $\frac{3\pi}{10}$.

2562. $\frac{\pi}{2}(15-16\ln 2)$. **2563.** $\pi\left(\frac{\pi^2}{4}-2\right)$.

2564. $\frac{8\pi}{3}$. **2565.** $2\pi^2$.

2566. $\frac{\pi a^3}{4}\left[\sqrt{2}\ln(1+\sqrt{2})+-\frac{2}{3}\right]$.

2567. 1) $\frac{2}{3}\pi a^3$; 2) $\frac{\pi^2}{16}$. **2568.** $5\pi^2a^3$.

2569. $\pi a^3\left(\frac{3\pi^2}{2}-\frac{8}{3}\right)$. **2570.** $\frac{32}{105}\pi a^3$.

2571. $\frac{16\pi c^6}{105ab^2}$. **2572.** $\frac{\pi^2}{2}$.

2573. $\frac{\pi e}{2}$.

2574*. 1) π; 2) $\pi\sqrt{\frac{\pi}{2}}$.

Vidi uputstvo uz zadatak 2516.

2575*. $\frac{3\pi\sqrt{2\pi}}{32}$.

Vidi uputstvo uz zadatak 2516.

Zadaci za vježbu

2576. Figura koju obrazuju kriva $y=\frac{\sin x}{x}$ i njena asimptota obrće se oko apscisne ose; izračunati zapreminu tako nastalog tela.

2577*. Izračunati zapreminu tela koje nastaje obrtanjem cisoide $y^2=\frac{x^3}{2a-x}$ $(a>0)$ oko njene asimptote.

2578. Naći zapreminu tela koje nastaje obrtanjem traktrise $x=a\left(\cos t+\ln \operatorname{tg}\frac{t}{2}\right)$, $y=a\sin t$ oko njene asimptote.

2579*. Izračunati zapreminu elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

2580. 1) Izračunati zapreminu tela ograničenog eliptičkim paraboloidom $z=\frac{x^2}{4}+\frac{y^2}{2}$ i ravni $z=1$.

2) Naći zapreminu tela ograničenog jednogranim hiperboloidom $\frac{x^2}{4}+\frac{y^2}{9}-z^2=1$ i ravnima $z=-1$ i $z=2$.

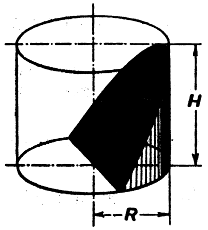
2581. Izračunati zapremine tela ograničenih paraboloidom $z=x^2+2y^2$ i elipsoidom $x^2+2y^2+z^2=6$.

2582. Naći zapremine tela ograničenih površinama dvogranog hiperboloida $\frac{x^2}{3}-\frac{y^2}{4}-\frac{z^2}{9}=1$ i elipsoida $\frac{x^2}{6}+\frac{y^2}{4}+\frac{z^2}{9}=1$.

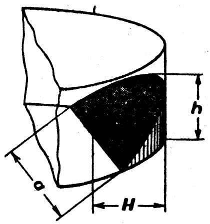
2583. Naći zapreminu tela ograničenog konusnom površinom $(z-2)^2=\frac{x^2}{3}+\frac{y^2}{2}$ i ravni $z=0$.

2584. Naći zapreminu tela ograničenog paraboloidom $2z=\frac{x^2}{4}+\frac{y^2}{9}$ i konusom $\frac{x^2}{4}+\frac{y^2}{9}=z^2$.

2585*. Od kružnog cilindra odsečen je jedan deo tako da presečna ravan prolazi kroz prečnik osnove cilindra („cilindrični odsečak“, sl. 43); naći zapreminu tog dela u opštem slučaju, i posebno za $R=10\ cm$, $H=6\ cm$.



Sl. 43



Sl. 44

2586. Parabolični cilindar presečen je dvema ravnima od kojih je jedna normalna na izvodnicu; tako obrazovano telo prikazano je na sl. 44. Zajednička osnovica paraboličnih odsečaka je $a=10\ cm$, visina paraboličnog odsečka koji leži u osnovi tela je $H=8\ cm$, a visina samog tela je $h=6\ cm$. Izačunati zapreminu tela.

2587. Eliptični cilindar presečen je tako da presečna ravan prolazi kroz malu osu; izračunati zapreminu odsečenog dela (potrebni numerički podaci o dimenzijama dati su na sl. 45).

Rješenja

2576*. π^2. Iskoristiti obrazac

$$\int_0^\infty \frac{\sin x}{x}\,dx=\frac{\pi}{2}.$$ (Dirihle-ov integral).

2577*. $2\pi^2a^3$. Preporučljivo je preći na parametarske jednačine:

$$x=2a\sin^2 t,\ y=\frac{2a\sin^3 t}{\cos t}.$$

2578. $\frac{2}{3}\pi a^3$. **2579*.** $\frac{4}{3}\pi abc$.

Primeniti obrazac

$$v=\int_{x_1}^{x_2} S(x)\,dx,$$ gde je $S(x)$ površina poprečnog preseka.

2580. 1) $\pi\sqrt{2}$; 2) 36π.

2581. $v_1=\pi\sqrt{2}\left(2\sqrt{6}-\frac{11}{3}\right)$,

$v_2=\pi\sqrt{2}\left(2\sqrt{6}+\frac{11}{3}\right)$.

2582. $v_1=v_3=4\pi(\sqrt{6}+\sqrt{3}-4)$,

$v_2=8\pi(4-\sqrt{3})$.

2583. $\frac{8\pi\sqrt{6}}{3}$. **2584.** 8π.

2585*. $\frac{2}{3}R^2H=400\ cm^3$.

Za apscisnu osu uzeti osu simetrije osnove.

2586. $\frac{4}{15}ahH=128\ cm^3$.

2587. $\frac{2}{3}abH=133\frac{1}{3}\ cm^3$.

Zadaci za vježbu

Površina obrtne površi

2594. Naći površinu površi koja nastaje kad se luk parabole $y^2=4ax$ od njenog temena do tačke sa apscisom $x=3a$, obrće oko apscisne ose.

2595. Izračunati površinu površi koja nastaje obrtanjem parabole trećeg stepena $3y-x^3=0$ oko apscisne ose (od $x_1=0$ do $x_2=a$).

2596. Izračunati površinu katenoida (uporedi i zadatak 2560) koji nastaje obrtanjem lančanice $y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right)$ oko apscisne ose (od $x_1=0$ do $x_2=a$).

2597. Obrtanjem elipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ oko velike ose dobija se izduženi obrtni elipsoid, a obrtanjem oko male ose — spljošteni obrtni elipsoid.); naći površinu i jednog i drugog.

2598. Izračunati površinu tela koje nastaje obrtanjem jednog svoda krive $y=\sin x$ oko apscisne ose.

2599. Luk krive $y=\operatorname{tg} x$ od tačke (0, 0) do tačke $\left(\frac{\pi}{4},\ 1\right)$ obrća se oko apscisne ose, izračunati površinu tako nastale površi.

2600. Naći površinu tela koje nastaje obrtanjem petlje krive $9ay^2=x(3a-x)^2$ oko apscisne ose.

2601. Luk kruga $x^2+y^2=a^2$ koji leži u prvom kvadrantu, obrće se oko svoje tetive; izračunati površinu tako nastalog tela.

2602. Naći površinu površi koja nastaje obrtanjem oko x-ose luka krive

$$x=e^t\sin t,\quad y=e^t\cos t\qquad \text{od } t_1=0 \text{ do } t_2=\frac{\pi}{2}$$

2603. Naći površinu tela koje nastaje obrtanjem astroide $x=a\cos^3 t$, $y=a\sin^3 t$ oko apscisne ose.

2604. Luk cikloide obrće se oko svoje ose simetrije; naći površinu tako nastale površi (vidi zad. 2568).

2605. Naći površinu tela koje nastaje obrtanjem kardioide $\rho=a(1+\cos\varphi)$ oko polarne ose.

2606. Krug $\rho=2r\sin\varphi$ obrće se oko polarne ose; naći površinu tako nastalog tela.

2607 Izračunati površinu tala koje nastaje obrtanjem lemniskate $\rho^2=a^2\cos 2\varphi$ oko polarne ose.

2608. Beskonačni luk krive $y=e^{-x}$ koji odgovara pozitivnim vrednostima promenljive x, obrće se oko x-ose; naći površinu tako nastale površi.

2609. Naći površinu beskrajnog vretena koje nastaje obrtanjem traktrise $x=a\left(\cos t+\ln \operatorname{tg}\frac{t}{2}\right)$, $y=a\cdot\sin t$ oko apscisne ose.

Rješenja

2594. $\frac{56}{3}\pi a^2$.

2595. $\frac{\pi}{9}\left(\sqrt{(1+a^4)^3}-1\right)$.

2596. $\frac{\pi a^2}{4}(e^2-e^{-2}+4)$.

(Vidi uputstvo uz zadatak 2588.)

2597. $2\pi b^2+\frac{2\pi ab}{\varepsilon}\arcsin\varepsilon$ i

$2\pi a^2+\frac{\pi b^2}{\varepsilon}\ln\frac{1+\varepsilon}{1-\varepsilon}$, gde je

ε — ekscentricitet elipse.

2598. $\pi[\sqrt{2}+\ln(1+\sqrt{2})]$.

2599. $\pi\left[\sqrt{5}-\sqrt{2}+\ln\frac{2\sqrt{2}+2}{\sqrt{5}+1}\right]$.

2600. $3\pi a^2$.

2601. $\pi a^2\sqrt{2}\left(2-\frac{\pi}{2}\right)$.

2602. $\frac{2\pi\sqrt{2}}{5}(e^{\pi}-2)$. **2603.** $\frac{12}{5}\pi a^2$.

2604. $8\pi a^2\left(\pi-\frac{4}{3}\right)$.

2605. $\frac{32}{5}\pi a^2$. **2606.** $4\pi^2r^2$.

2607. $2\pi a^2(2-\sqrt{2})$.

2608. $\pi[\sqrt{2}+\ln(1+\sqrt{2})]$.

2609. $4\pi a^2$.

Funkcija dvije nezavisne promjenjive

Neka je S neprazan podskup prostora $\mathbb{R}^2$ i $T \subseteq \mathbb{R}$. Ako svakoj tački $M(x,y) \in S$ možemo unaprijed po datom pravilu f pridružiti jednu i samo jednu realnu vrijednost $z \in T$, tada kažemo da je data realna f-ja dvije realne promjenjive f iz $\mathbb{R}^2$ u $\mathbb{R}$ (sa skupa $S \subseteq \mathbb{R}$ u skup $T \subseteq \mathbb{R}$) i pišemo $z = f(x,y)$. Skup S na kojem je određena f-ja f naziva se domen ili definiciono područje f-je f (označavat ćemo ga sa $\mathcal{D}(f)$), a skup f(A) skup vrijednosti f-je f ili kodomen (označavat ćemo ga sa $\mathcal{R}(f)$. Ako za f-ju, zadanu analitički (formulom) nije data oblast njene definisanosti, onda se pod njom podrazumjeva skup svih tačaka $M \in \mathbb{R}^2$ u kojoj f-ja, odnosno njen analitički izraz imaju određenu realnu vrijednost.

(#) Za svaku od sljedećih f-ja, izračunati f(3,2), i odrediti iskicirati domen.

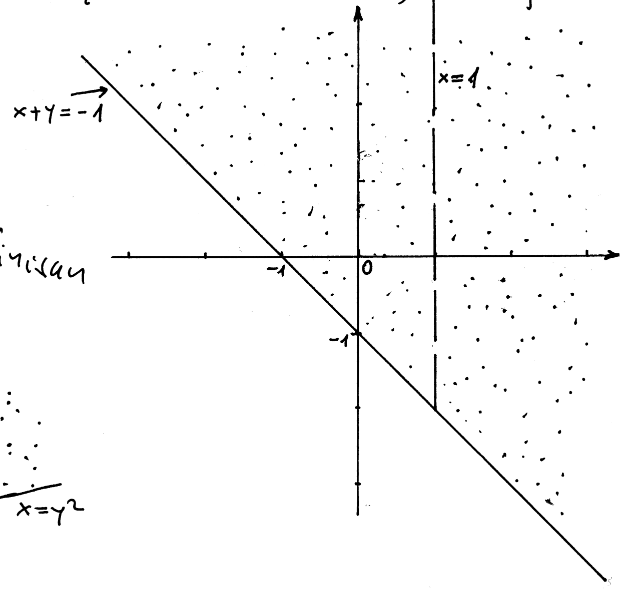
a) $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$ b) $f(x,y) = x\ln(y^2 - x)$

Rj. a) $f(3,2) = \dfrac{\sqrt{3+2+1}}{3-1} = \dfrac{\sqrt{6}}{2}$

Izraz za f-ju f(x,y) ima smisla ako je nazivnik različit od nule i ako je vrijednost pod korijenom nenegativna:

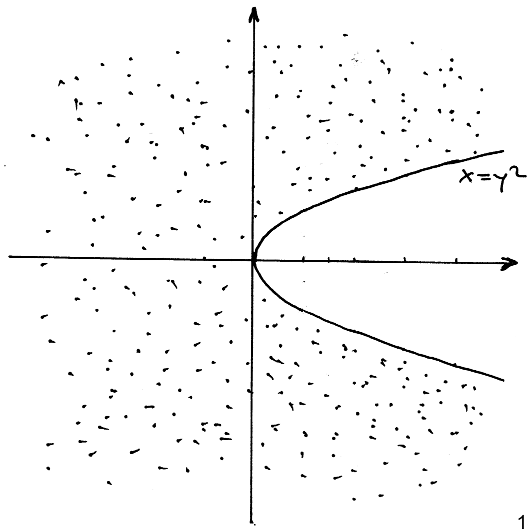
$$\begin{matrix} x-1 \neq 0 \\ x+y+1 \geq 0 \end{matrix} \Rightarrow \begin{matrix} x \neq 1 \\ x+y \geq -1 \end{matrix}$$

Domen f-je f je $D = \{(x,y) \in \mathbb{R}^2 \mid x+y \geq -1,\ x \neq 1\}$



b) $f(3,2) = 3\ln(2^2-3)$
$= 3\ln(4-3) = 3\underbrace{\ln 1}_{=0} =$
$= 0$

Izraz $\ln(y^2 - x)$ je definisan samo ako je $y^2 - x > 0$



$D = \{(x,y) \mid x < y^2\}$

(#) Odrediti domen i rang f-je $g(x,y)=\sqrt{9-x^2-y^2}$

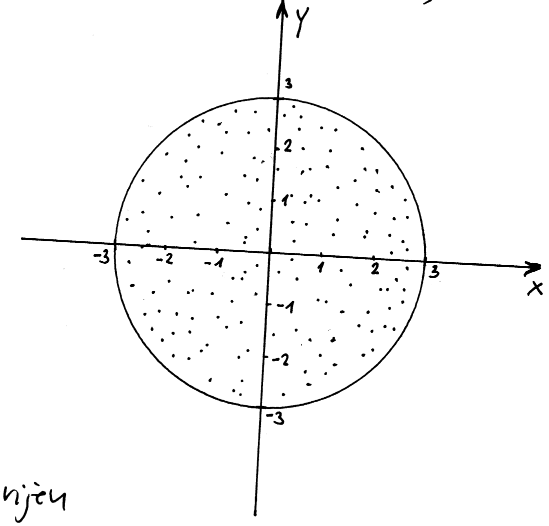
Rj.

F-ja ima smisla akko $9-x^2-y^2 \geq 0$

$$x^2+y^2 \leq 9$$

Domen f-je g(x,y) je $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 9\}$

(znamo da je $x^2+y^2=9$ krug sa centrom u tački C(0,0) poluprečnika r=3).



Rang f-je g je

$\{z\in\mathbb{R} \mid z=\sqrt{9-x^2-y^2},\ (x,y)\in D\}$

Primjetimo da je

$9-x^2-y^2\leq 9$ za $\forall (x,y)\in D$

pa je $\sqrt{9-x^2-y^2}\leq 3$

z je pozitivan kvadratni korijen

$$z\geq 0$$

Prema tome, rang f-je g(x,y) je

$$\{z \mid 0\leq z\leq 3\} = [0,3]$$

(#) Skicirati graf f-je $f(x,y)=6-3x-2y$.

Rj.

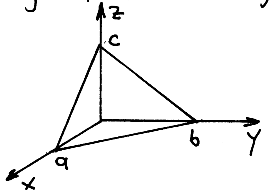
Graf f-je f(x,y) ima jednačinu $z=6-3x-2y$

$$3x+2y+z=6$$

ovo predstavlja ravan.

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

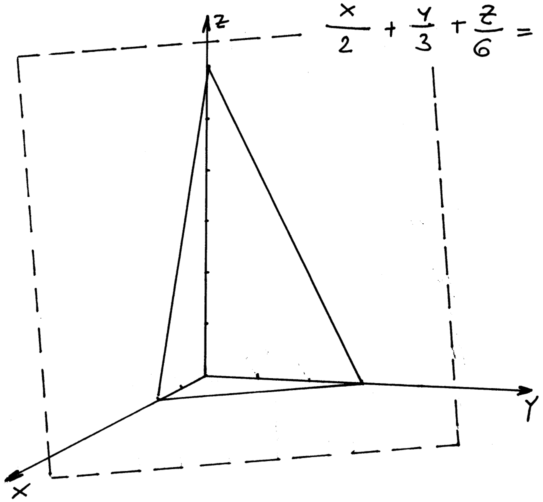
segmentni oblik jednačine ravni



U našem slučaju

$3x+2y+z=6 \quad /:6$

$\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$



(#) Skicirati graf f-je $g(x,y) = \sqrt{9-x^2-y^2}$.

Rj. Graf f-je ima jednačinu $z = \sqrt{9-x^2-y^2}$

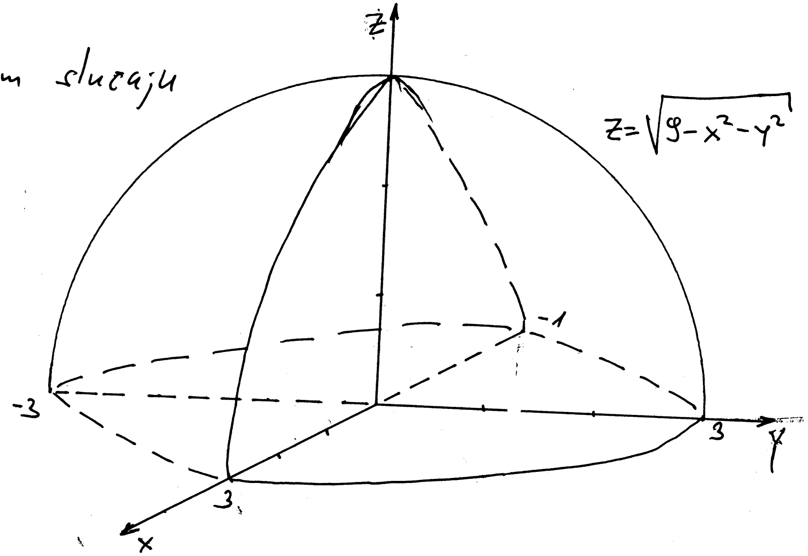
$$z = \sqrt{9-x^2-y^2} \quad /^2$$

$$z^2 = 9 - x^2 - y^2$$

$$x^2 + y^2 + z^2 = 9$$

$x^2+y^2+z^2 = R^2$ je jednačina sfere sa centrom u koordinatnom početku poluprečnika R

U našem slučaju



Zadaci za vježbu

§ 2. Početno proučavanje funkcije

Oblast definisanosti

2975. Oblast koja leži unutar paralelograma, obrazovanog pravama: $y=0$, $y=2$, $y=\frac{1}{2}x$, $y=\frac{1}{2}x-1$ prikazati pomoću nejednakosti.

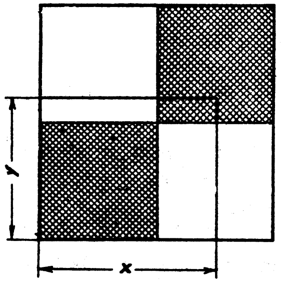
2976. Oblast ograničenu parabolama $y=x^2$ i $x=y^2$ (uključujući granice) definisati nejednakostima.

2977. Opisati pomoću nejednakosti otvorenu oblast, ograničenu jednakostraničnim trouglom stranice a, sa jednim temenom u koordinatnom početku, drugim — na pozitivnom delu x-ose, i trećim — u prvom kvadrantu.

2978. Oblast je ograničena beskonačnim kružnim cilindrom poluprečnika R (isključujući granice), čija je osa paralelna z-osi i prolazi kroz tačku (a, b, c); opisati ovu oblast pomoću nejednakosti.

2979. Oblast ograničenu sferom poluprečnika R sa centrom u tački (a, b, c) (uključujući granicu) definisati pomoću nejednakosti.

2980. Temena pravouglog trougla leže unutar kruga poluprečnika R. Površina S trougla je funkcija njegovih kateta x i y: $S=\varphi(x, y)$: naći: a) oblast definisanosti funkcije φ; b) oblast definisanosti odgovarajućeg analitičkog izraza.



Sl. 57

2981. U loptu poluprečnika R upisana je prava piramida sa pravougaonikom u osnovi. Zapremina V piramide je funkcija osnovnih ivica x i y. Hoće li ova funkcija biti jednoznačno definisana? Sastaviti njoj odgovarajući analitički izraz. i naći oblast definisanosti funkcije i pomenutog analitičkog izraza.

2982. Kvadratna daska se sastoji iz četiri kvadratna polja, dva crna i dva bela kao što je to prikazano na sl. 57; stranica svakog od njih ima dužinu 1. Uočimo pravougaonik čije su stranice x i y paralelne stranicama daske i čiji se jedan ugao poklapa sa njenim crnim uglom. Površina crnog dela ovog pravougaonika biće funkcija od x i y. Naći oblast definisanosti ove funkcije. Izraziti ovu funkciju analitički.

U zadacima 2983—3002 naći oblast definisanosti datih funkcija

2983. $z=\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$.

2984. $z=\ln(y^2-4x+8)$.

2985. $z=\frac{1}{R^2-x^2-y^2}$.

2986. $z=\sqrt{x+y}+\sqrt{x-y}$.

2987. $z=\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{x-y}}$.

2988. $z=\arcsin\frac{y-1}{x}$.

2989. $z=\ln xy$.

2990. $z=\sqrt{x-\sqrt{y}}$.

2991. $z=\arcsin\frac{x^2+y^2}{4}+\operatorname{arcsec}(x^2+y^2)$.

2992. $z=\frac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$.

2993. $z=\sqrt{\frac{x^2+2x+y^2}{x^2-2x+y^2}}$.

2994. $z=xy+\sqrt{\ln\frac{R^2}{x^2+y^2}}+\sqrt{x^2+y^2-R^2}$.

2995. $z=\operatorname{ctg}\pi(x+y)$.
2996. $z=\sqrt{\sin\pi(x^2+y^2)}$.

2997. $z=\sqrt{x\sin y}$.
2998. $z=\operatorname{lm}x-\ln\sin y$.

2999. $z=\ln[x\ln(y-x)]$.
3000. $z=\arcsin[2y(1+x^2)-1]$.

Rješenja

2975. $0<y<2;\ -1<y-\frac{1}{2}x<0$. **2976.** $x^2\leqslant y\leqslant\sqrt{x}$.

2977. $0<y<x\sqrt{3};\ y<(a-x)\sqrt{3}$. **2978.** $(x-a)^2+(y-b)^2<R^2;\ -\infty<z<\infty$.

2979. $(x-a)^2+(y-b)^2+(z-c)^2\leqslant R^2$. **2980.** a) $x^2+y^2\leqslant 4R^2$; b) $-\infty<x<\infty;\ -\infty<y<\infty$.

2981. $v=\frac{1}{6}xy(2R\pm\sqrt{4R^2-x^2-y^2})$; funkcija nije jednoznačna. Oblast definisanosti funkcije je $x^2+y^2\leqslant 4R^2$; $x>0$, $y>0$. Oblast definisanosti analitičkog izraza je $x^2+y^2\leqslant 4R^2$.

2982. Za $0\leqslant x\leqslant 1$, $0\leqslant y\leqslant 1$ $S=xy$; za $1\leqslant x\leqslant 2$, $2\leqslant y$ $S=x$;
za $0\leqslant x\leqslant 1$, $1\leqslant y$ $S=x$; za $2\leqslant x$, $1\leqslant y\leqslant 2$ $S=y$;
za $1\leqslant x$ $0\leqslant y\leqslant 1$ $S=y$; za $2\leqslant x$, $2\leqslant y$ $S=2$;
za $1\leqslant x\leqslant 2$, $1\leqslant y\leqslant 2$ $S=xy-x-y+2$; funkcija nije definisana za $x<0$ i $y<0$.

2983. $\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1$. **2984.** $y^2>4x-8$.

2985. Sva ravan izuzev tačaka kružne linije $x^2+y^2=R^2$.

2986. Unutrašnjost desnog pravog ugla koji obrazuju simetrale koordinatnih uglova, uključujući i odgovarajuće delove simetrale, tj.

$$x+y\geqslant 0,\ x-y\geqslant 0.$$

2987. Ista kao i u zad. 2986, samo bez tačaka na granici oblasti.

2988. Unutrašnjost desnog i levog ugla koje obrazuju prave $y=1+x$ i $y=1-x$, uključujući i te prave, ali bez njihove presečne tačke:

$1-x\leqslant y\leqslant 1+x$ $(x>0)$, $1+x\leqslant y\leqslant 1-x$ $(x<0)$. (za $x=0$ funkcija nije definisana).

2989. Unutrašnjost prvog i trećeg kvadranta.

2990. Zatvorena oblast između pozitivnog dela apscisne ose i parabole $y=x^2$ (isključujući i granicu):

$$x\geqslant 0,\ y\geqslant 0;\ x^2\geqslant y.$$

2991. Prstenasta oblast između krugova $x^2+y^2=1$ i $x^2+y^2=4$, uključujući i samo krugove: $1\leqslant x^2+y^2<4$.

2992. Deo ravni koji leži unutar parabole $y^2=4x$, između parabole i kruga $x^2+y^2=1$, uključujući luk parabole izuzev njegovog temena, i isključujući luk kruga.

2993. Deo ravni koji leži izvan krugova čiji su poluprečnici jednaki jedinici a centri su im u tačkama $(-1, 0)$ i $(1, 0)$; tačke prvog kruga pripadaju oblasti, tačke drugog ne pripadaju.

2994. Samo tačke kružne linije $x^2+y^2=R^2$.

2995. Sva ravan, izuzev pravih $x+y=n$ (n je ma koji ceo broj, pozitivan, negativan ili nula).

2996. Unutrašnjost kruga $x^2+y^2=1$ i prsten $2n\leqslant x^2+y^2\leqslant 2n+1$ (n je ceo broj), uključujući i granice.

2997. Ako je $x\geqslant 0$, onda je $2n\pi\leqslant y\leqslant(2n+1)\pi$, ako je $x<0$, onda je $(2n+1)\pi\leqslant y\leqslant(2n+2)\pi$, pri čemu je n ceo broj.

2998. $x>0$; $2n\pi<y<2(n+1)\pi$ (n je ceo broj).

2999. Otvorena šrafirana oblast prikazana na sl. 83: za $x>0$ je $y>x+1$; za $x<0$ je $x<y<x+1$.

Parcijalni izvodi f-ja više promjenjivih

Posmatrajmo f-ju z dvije promjenjive $z=f(x,y)$.
Parcijalni izvod po x-u označavamo sa z'_x ili sa $\frac{\partial z}{\partial x}$ (delta z po delta x) ili sa f'_x i definišemo

$$z'_x=\frac{\partial z}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

Parcijalni izvod po y-nu označavamo sa z'_y ili sa $\frac{\partial z}{\partial y}$ (∂ - delta) ili sa f'_y i definišemo

$$z'_y=\frac{\partial z}{\partial y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}$$

Odrediti parcijalne izvode f-ja

a) $z = x^3 + 5xy^2 - y^3$

b) $u = \frac{x}{y} + \frac{y}{z} - \frac{z}{x}$

c) $v = \sqrt[x]{e^y}$

Rj. a) Kad vadimo izvod po x-u, samo x tumačimo kao promjenjivu, sve ostalo tumačimo kao broj.

$$\frac{\partial z}{\partial x} = 3x^2 + 5y^2 .$$

Analogno za y-om $\quad \frac{\partial z}{\partial y} = 10xy - 3y^2 .$

b)
$$\frac{\partial u}{\partial x} = \frac{1}{y} - z \cdot \left(\frac{1}{x}\right)'_x = \frac{1}{y} - z \cdot (-1)x^{-2} = \frac{1}{y} + \frac{z}{x^2}$$

$$\frac{\partial u}{\partial y} = x \cdot (-1) y^{-2} + \frac{1}{z} = -\frac{x}{y^2} + \frac{1}{z}$$

$$\frac{\partial u}{\partial z} = y \cdot \left(\frac{1}{z}\right)'_z - \frac{1}{x} = y \cdot (-1) z^{-2} - \frac{1}{x} = -\frac{y}{z^2} - \frac{1}{x}$$

c)
$$\frac{\partial v}{\partial x} = \left(e^{\frac{y}{x}}\right)'_x = e^{\frac{y}{x}} \cdot \left(\frac{y}{x}\right)'_x = y e^{\frac{y}{x}} \cdot (x^{-1})'_x = -y e^{\frac{y}{x}} \cdot x^{-2} = -\frac{y}{x^2} e^{\frac{y}{x}}$$

$$\frac{\partial v}{\partial y} = \left(e^{\frac{y}{x}}\right)'_y = e^{\frac{y}{x}} \cdot \left(\frac{y}{x}\right)'_y = \frac{1}{x} e^{\frac{y}{x}}$$

Pronaći vrijednost parcijalnih izvoda datih f-ja u datim tačkama

a) $f(\alpha, \beta) = \cos(m\alpha - n\beta), \quad \alpha = \frac{\pi}{2m}, \quad \beta = 0;$

b) $z = \ln(x^2 - y^2), \quad x = 2, \quad y = -1.$

Rj. a)
$$f'_\alpha = -\sin(m\alpha - n\beta) \cdot (m\alpha - n\beta)'_\alpha = -m \sin(m\alpha - n\beta)$$

$$f'_\beta = -\sin(m\alpha - n\beta) \cdot (m\alpha - n\beta)'_\beta = n \sin(m\alpha - n\beta)$$

$$f'_\alpha\left(\frac{\pi}{2m}, 0\right) = -m \sin\frac{\pi}{2} = -m, \quad f'_\beta\left(\frac{\pi}{2m}, 0\right) = n \sin\frac{\pi}{2} = n$$

b)
$$z'_x = \frac{1}{x^2 - y^2} \cdot 2x$$

$$z'_y = \frac{1}{x^2 - y^2} \cdot (-2y)$$

$$z'_x(2, -1) = \frac{1}{4-1} \cdot 2$$

$$z'_y(2, -1) = \frac{1}{4-1} \cdot 2 = \frac{2}{3}$$

㊉ Naći sve parcijalne izvode prvog reda f-je

a) $z=x^2y^5+3x^3y-7$ c) $z=(2x^2y^2-x+1)^3$ e) $z=\operatorname{arctg}\frac{y}{x}$

b) $z=x^y$ d) $z=\frac{x+y^2}{x^2+y^2+1}$ f) $u=\sqrt{x^2+y^2+z^2}$

g) $u=\ln(x^3-y^2+z^4)$

Rj.

a) $z'_x=2xy^5+9x^2y$

$z'_y=x^2\cdot 5y^4+3x^3=5x^2y^4+3x^3$

b) $z'_x=yx^{y-1}$

$z'_y=x^y\ln x$

c) $z'_x=3(2x^2y^2-x+1)^2(4xy^2-1)$

$z'_y=3(2x^2y^2-x+1)^2(4x^2y)=12x^2y(2x^2y^2-x+1)^2$

d)

$$z'_x=\frac{1\cdot(x^2+y^2+1)-(x+y^2)\cdot 2x}{(x^2+y^2+1)^2}=\frac{x^2+y^2+1-2x^2-2xy^2}{(x^2+y^2+1)^2}=\frac{-x^2+y^2+1-2xy^2}{(x^2+y^2+1)^2}$$

$$z'_y=\frac{2y(x^2+y^2+1)-(x+y^2)(2y)}{(x^2+y^2+1)^2}=\frac{2x^2y\cancel{+2y^3}+2y-2xy\cancel{-2y^3}}{(x^2+y^2+1)^2}=\frac{2y(x^2-x+1)}{(x^2+y^2+1)^2}$$

e) $z=\operatorname{arctg}\frac{y}{x}$

$$z'_x=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{y}{x}\right)'_x=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)=\frac{(-1)\cdot y}{\left(1+\frac{y^2}{x^2}\right)\cdot x^2}=\frac{-y}{x^2+y^2}$$

$$z'_y=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}=\frac{1}{\left(1+\frac{y^2}{x^2}\right)\cdot x}=\frac{x}{x^2+y^2}$$

f) $u=\sqrt{x^2+y^2+z^2}$, $u'_x=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x=\frac{x}{\sqrt{x^2+y^2+z^2}}$

$u'_y=\frac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2y=\frac{y}{\sqrt{x^2+y^2+z^2}}$, $u'_z=\frac{z}{\sqrt{x^2+y^2+z^2}}$

g) $u=\ln(x^3-y^2+z^4)$, $u'_x=\frac{3x^2}{x^3-y^2+z^4}$, $u'_y=\frac{-2y}{x^3-y^2+z^4}$, $u'_z=\frac{4z^3}{x^3-y^2+z^4}$

㊉ Proveriti da li f-ja $z=x\ln\frac{y}{x}$ zadovoljava jednakost

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$$

Rj.

$$\frac{\partial z}{\partial x}=1\cdot\ln\frac{y}{x}+x\cdot\frac{1}{\frac{y}{x}}\cdot\left(\frac{y}{x}\right)'_x=\ln\frac{y}{x}+\frac{x^2}{y}\cdot(-1)y(x)^{-2}=\ln\frac{y}{x}-1$$

F-ju z možemo napisati i u obliku $z=x(\ln y-\ln x)$

$$\frac{\partial z}{\partial y}=x\cdot\frac{1}{y}=\frac{x}{y}$$

$$x\cdot\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=x\left(\ln\frac{y}{x}-1\right)+y\cdot\frac{x}{y}=x\ln\frac{y}{x}-x+x=x\ln\frac{y}{x}=z$$

F-ja $z=x\ln\frac{y}{x}$ zadovoljava datu jednakost.

㊉ Ako je $z=x^y\cdot y^x$ dokazati da je

$$x\cdot\frac{\partial z}{\partial x}+y\cdot\frac{\partial z}{\partial y}=z\cdot(x+y+\ln z).$$

Rj.

$$\frac{\partial z}{\partial x}=yx^{y-1}\cdot y^x+x^y\cdot y^x\ln y$$

$$\frac{\partial z}{\partial y}=x^y\ln x\cdot y^x+x^y\cdot xy^{x-1}$$

$$x\cdot\frac{\partial z}{\partial x}=xy\,x^{y-1}y^x+x\ln y\,x^yy^x$$

$$=y\,x^yy^x\underline{+x\ln y\,x^yy^x}$$

$$y\cdot\frac{\partial z}{\partial y}=\underline{y\ln x\cdot x^yy^x}+x\cdot x^yy^x$$

$$x\cdot\frac{\partial z}{\partial x}+y\cdot\frac{\partial z}{\partial y}=yx^yy^x+\ln y^x\cdot x^yy^x+x^yy^x\ln x^y+x\,x^yy^x=$$

$$=x^yy^x\left(y+\ln(x^y\cdot y^x)+x\right)=z\cdot(x+y+\ln z)$$

što je i trebalo dobiti

Zadaci za vježbu

Naći parcijalne izvode sledećih f-ja

① $z=(5x^3y^3+1)^3$ ② $r=\sqrt{ax^2-by^2}$

③ $v=\ln(x+\sqrt{x^2+y^2})$ ④ $\rho=\arcsin\frac{x}{t}$

⑤ $f(m,n)=(2m)^{3n}$; izračunati f'_m i f'_n u tački $A(\frac{1}{2};2)$

⑥ $\rho(x,y,z)=\sin^2(3x+2y-z)$; izračunati $\rho'_x(1;-1;1)$, $\rho'_y(1;1;4)$, $\rho'_z(-\frac{1}{2};0;-1)$

⑦ Provjeriti da li f-ja $v=x^y$ zadovoljava jednakost

$$\frac{x}{y}\cdot\frac{\partial v}{\partial x}+\frac{1}{\ln x}\cdot\frac{\partial v}{\partial y}=2v$$

⑧ Provjeriti da li f-ja $w=x+\frac{x-y}{y-z}$ zadovoljava jednakost

$$\frac{\partial w}{\partial x}+\frac{\partial w}{\partial y}+\frac{\partial w}{\partial z}=1.$$

Rješenja:

1. $z'_x=45x^2y^2(5x^3y^2+1)^2$;

$z'_y=30x^3y(5x^3y^2+1)^2$.

2. $\frac{\partial r}{\partial x}=\frac{ax}{r}$; $\frac{\partial r}{\partial y}=-\frac{by}{r}$.

3. $\frac{\partial v}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}$;

$\frac{\partial v}{\partial y}=\frac{y}{(x+\sqrt{x^2+y^2})\sqrt{x^2+y^2}}$.

4. $\frac{\partial \rho}{\partial x}=\frac{|t|}{t\sqrt{t^2-x^2}}$;

$\frac{\partial \rho}{\partial t}=-\frac{x}{|t|\sqrt{t^2-x^2}}$.

5. 12; 0.

6. 0; 2 sin 2; − sin (−1)

Diferenciranje f-ja više promjenjivih

Posmatrajmo f-ju tri promjenjive $u=f(x,y,z)$. Diferencijal f-je u označavamo sa du i računamo po formuli

$$du=d_xu+d_yu+d_zu=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$

gdje su d_xu, d_yu, d_zu parcijalni diferencijali f-je u redom po promjenjivim x, y i z.

$$d_xu=\frac{\partial u}{\partial x}dx,\quad d_yu=\frac{\partial u}{\partial y}dy,\quad d_zu=\frac{\partial u}{\partial z}dz.$$

(#) Odrediti totalne diferencijale f-ja

a) $z=3x^2y^5$ b) $u=2x^{yz}$ c) $p=\arccos\frac{1}{uv}$

Rj.

a) Parcijalni izvodi su

$$\frac{\partial z}{\partial x}=6xy^5, \qquad \frac{\partial z}{\partial y}=15x^2y^4$$

Totalni diferencijal je $dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$ tj.

$$dz=6xy^5dx+15x^2y^4dy$$

b) Parcijalni izvodi su

$$\frac{\partial u}{\partial x}=2yzx^{yz-1}, \qquad \frac{\partial u}{\partial y}=2x^{yz}\ln x\cdot z, \qquad \frac{\partial u}{\partial z}=2yx^{yz}\ln x$$

Totalni diferencijal je

$$du=2yzx^{yz-1}dx+2zx^{yz}\ln x\,dy+2yx^{yz}\ln x\,dz$$

$$=2x^{yz}\left(\frac{yz}{x}dx+z\ln x\,dy+y\ln x\,dz\right)$$

c) Parcijalni izvodi su

$$\frac{\partial p}{\partial u}=\frac{-1}{\sqrt{1-\left(\frac{1}{uv}\right)^2}}\cdot\left(\frac{1}{uv}\right)'_u=\frac{-1}{\sqrt{\frac{u^2v^2-1}{u^2v^2}}}(-1)(uv)^{-2}\cdot v=\frac{|uv|}{u^2v\sqrt{u^2v^2-1}};$$

$$\frac{\partial p}{\partial v}=\frac{-1}{\sqrt{1-\frac{1}{u^2v^2}}}(-1)(uv)^{-2}\cdot u=\frac{u}{\sqrt{\frac{u^2v^2-1}{u^2v^2}}}\cdot\frac{1}{u^2v^2}=\frac{|uv|}{uv^2\sqrt{u^2v^2-1}};$$

Totalni diferencijal

$$dp=\frac{1}{\sqrt{u^2v^2-1}}\left(\frac{|uv|}{u^2v}du-\frac{|uv|}{uv^2}dv\right)=\frac{1}{\sqrt{u^2v^2-1}}\left(\frac{|v|\,du}{v\,|u|}-\frac{|u|\,dv}{u|v|}\right).$$

(#) Odrediti parcijalne diferencijale f-je $z=\sqrt[3]{x^3+y^3}$.

Rj.

$$z'_x=\frac{\partial z}{\partial x}=\frac{1}{3}(x^3+y^3)^{-\frac{2}{3}}\cdot 3x^2=\frac{x^2}{\sqrt[3]{(x^3+y^3)^2}}$$

$$z'_y=\frac{\partial z}{\partial y}=\frac{1}{3}(x^3+y^3)^{-\frac{2}{3}}\cdot 3y^2=\frac{y^2}{\sqrt[3]{(x^3+y^3)^2}}$$

dobijeni izrazi za parcijalne izvode nisu definisani u tački (0,0). Izvode u toj tački treba odrediti po definiciji

$$z'_x(0,0)=\lim_{\varepsilon\to 0}\frac{z(0+\varepsilon,0)-z(0,0)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{\sqrt[3]{\varepsilon^3+0^3}-0}{\varepsilon}=\lim_{\varepsilon\to 0}1=1$$

$$z'_y(0,0)=\lim_{\varepsilon\to 0}\frac{z(0,0+\varepsilon)-z(0,0)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{\sqrt[3]{0^3+\varepsilon^3}-0}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{\varepsilon}{\varepsilon}=1$$

f-ja f ima parcijalne izvode u svim tačkama iz oblasti definisanosti. Parcijalni diferencijali su

$$d_xz=\frac{\partial z}{\partial x}dx=\begin{cases}\frac{x^2}{\sqrt[3]{(x^3+y^3)^2}}dx, & (x,y)\neq(0,0)\\ dx, & (x,y)=(0,0)\end{cases}$$

$$d_yz=\frac{\partial z}{\partial y}dy=\begin{cases}\frac{y^2}{\sqrt[3]{(x^3+y^3)^2}}dy, & (x,y)\neq(0,0)\\ dy, & (x,y)=(0,0)\end{cases}$$

(#) Odrediti totalni diferencijal f-je $z=\arcsin\frac{x}{y}$ u tački (4,5).

Rj. F-ja je definisana za $\left|\frac{x}{y}\right|<1$

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(\frac{x}{y}\right)'_x=\frac{1}{y\sqrt{1-\frac{x^2}{y^2}}}=\frac{1}{\sqrt{y^2-x^2}},\quad \frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right)=\frac{-x}{y\sqrt{y^2-x^2}}$$

$$dz=\frac{1}{\sqrt{y^2-x^2}}dx+\frac{-x}{y\sqrt{y^2-x^2}}dy=\frac{y\,dx-x\,dy}{y\sqrt{y^2-x^2}}$$

Stavljajući u dobijeni izraz $x=4$ i $y=5$ dobijemo $dz=\frac{1}{15}(5dx-4dy)$

(#) Pomoću totalnog diferencijala približno izračunati $\ln(\sqrt[3]{1,03}+\sqrt[4]{0,98}-1)$.

Rj. Neka je $z=\ln(\sqrt[3]{x}+\sqrt[4]{y}-1)$ gdje je $x=a+\epsilon=1+0,03$ i $y=b+\omega=1-0,02$.
Tada je $z(a,b)=\ln(\sqrt{1}-\sqrt{1}-1)=\ln 1=0$; $z=z(a,b)+\Delta z$.
($\Delta z=f(a+\epsilon,b+\omega)-f(a,b)$ totalni priraštaj f-je u tački (a,b)).
Kako je $\Delta z\approx dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{1}{\sqrt[3]{x}+\sqrt[4]{y}-1}\left(\frac{1}{3\sqrt[3]{x^2}}dx+\frac{1}{4\sqrt[4]{y^3}}dy\right)=$
$=\frac{1}{1}\left(\frac{1}{3}\cdot 0,03-\frac{1}{4}\cdot 0,02\right)=0,005$. Pa $z=z_0+\Delta z\approx 0,0005$.

(#) Naći totalni diferencijal i totalni priraštaj f-je $z=x^2+y^2+xy$ pri prelazu od tačke (1,1) u tačku (1,1; 0,9).

Rj. po definiciji totalnog priraštaja dobijemo

$$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)=(x+\Delta x)^2+(y+\Delta y)^2+(x+\Delta x)(y+\Delta y)-(x^2+y^2+xy)=$$

$$=\underline{x^2}+\underset{\cdots\cdots}{2x\Delta x}+\underset{\cdots}{\Delta x^2}+\underline{\underline{y^2}}+2y\Delta y+\Delta y^2+\underline{\underline{\underline{xy}}}+x\Delta y+\underset{\cdots\cdots}{y\Delta x}+\Delta x\Delta y\underline{-x^2}\underline{\underline{-y^2}}\underline{\underline{\underline{-xy}}}$$

$$=2x\Delta x+\Delta x^2+y\Delta x+2y\Delta y+\Delta y^2+x\Delta y+\Delta x\Delta y=(2x+y+\Delta x)\Delta x+(2y+x+\Delta x+\Delta y)\Delta y$$

Ako stavimo u formulu vrijednosti $x=1$, $y=1$, $\Delta x=1,1-1=0,1$, $\Delta y=0,9-1=-0,1$ dobićemo totalni priraštaj date f-je u tački (1,1)

$$\Delta z=(2+1+0,1)0,1+(2+1+0,1-0,1)\cdot\cancel{(-0,1)}=3,1\cdot 0,1+3\cdot(-0,1)=0,31-0,3=0,01$$

$$dz=(2x+y)dx+(2y+x)dy\qquad dz=(2+1)\cdot 0,1+(2+1)\cdot(-0,1)=0,3-0,3=0$$

Diferenciranje složenih f-ja

F-ju z nazivamo složenom f-jom od tri nezavisno promjenjive x, y, t ako je ona zadana putem argumenata u, v, ..., w :

$$z=F(u,v,\dots,w)$$

gdje je

$$u=f(x,y,t),\quad v=\varphi(x,y,t),\quad\dots,\quad w=\psi(x,y,t).$$

Slično bi definisali f-ju od n nezavisno promjenjivih.

Parcijalni izvod složene f-je po jednoj od nezavisnih promjenjivih jednak je sumi proizvoda parcijalnog izvoda f-je po njenom argumentu sa parcijalnim izvodom istog argumenta po nezavisnoj promjenjivoj:

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}+\dots+\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial x};$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}+\dots+\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial y};\qquad\dots(*)$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial t}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial t}+\dots+\frac{\partial z}{\partial w}\cdot\frac{\partial w}{\partial t}.$$

Ako su svi argumenti u, v, ..., w f-je jedne nezavisno promjenjive x, tada je i z složena f-ja po promjenjivoj x. Izvod takve složene f-je (od jedne nezavisno promjenjive) naziva se <u>totalni izvod</u> i dat je preko formule

$$\frac{dz}{dx}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}+\dots+\frac{\partial z}{\partial w}\cdot\frac{dw}{dx}.\qquad\dots(**)$$

(dobije se iz formule totalnog diferencijala f-je $z(u,v,w)$ tako što je podjelimo sa dx).

(#) Naći izvode složenih f-ja

a) $y = u^2 e^v, \ u = \sin x, \ v = \cos x$;

b) $p = u^v, \ u = \ln(x-y), \ v = e^{\frac{x}{y}}$;

c) $z = x \sin v \cos w, \ v = \ln(x^2+1), \ w = -\sqrt{1-x^2}$.

Rj. a) Primjetimo da je y složena f-ja po nezavisno promjenjivoj x. Koristimo formulu (**)

$$\frac{dy}{dx} = \frac{\partial y}{\partial u} \cdot \frac{du}{dx} + \frac{\partial y}{\partial v} \cdot \frac{dv}{dx} \overset{(\square)}{=} 2ue^v \cos x - u^2 e^v \sin x$$

$$\frac{\partial y}{\partial u} = 2ue^v, \quad \frac{du}{dx} = \cos x, \quad \frac{\partial y}{\partial v} = u^2 e^v, \quad \frac{dv}{dx} = -\sin x \qquad \ldots(\square)$$

b) p je složena f-ja dvije promjenjive x, y. Koristimo formulu (*)

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial p}{\partial v} \cdot \frac{\partial v}{\partial x} = vu^{v-1} \cdot \frac{1}{x-y} + u^v \ln u \cdot \frac{1}{y} e^{\frac{x}{y}}$$

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial p}{\partial v} \cdot \frac{\partial v}{\partial y} = vu^{v-1} \cdot \frac{1}{y-x} + u^v \ln u \left(-\frac{x}{y^2} e^{\frac{x}{y}}\right)$$

c) z je složena f-ja jedne promjenjive x. Koristimo formulu (**).

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dx} + \frac{\partial z}{\partial w} \cdot \frac{dw}{dx}$$

$$\frac{dz}{dx} = \sin v \cos w + x \cos v \cos w \cdot \frac{2x}{x^2+1} - x \sin v \sin w \ \frac{x}{\sqrt{1-x^2}}$$

(#) Naći diferencijal f-je u (naći du) ako je $u = f(\sqrt{x^2+y^2})$.

Rj. $u = f(\sqrt{x^2+y^2})$, uvedimo oznaku $t = \sqrt{x^2+y^2}$.

$$u = f(t) = f(t(x,y)), \qquad du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$\frac{\partial u}{\partial x} = f'_t \cdot \frac{\partial t}{\partial x} = f'_t \ \frac{2x}{2\sqrt{x^2+y^2}} = \frac{x f'_t}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = f'_t \cdot \frac{\partial t}{\partial y} = f'_t \cdot \frac{2y}{2\sqrt{x^2+y^2}} = \frac{y \cdot f'_t}{\sqrt{x^2+y^2}}$$

$$du = \frac{f'_{\sqrt{x^2+y^2}}(\sqrt{x^2+y^2})(x\,dx + y\,dy)}{\sqrt{x^2+y^2}}$$

(#) Ako je $z = \dfrac{y}{f(x^2-y^2)}$ tada je $\dfrac{1}{x} \cdot \dfrac{\partial z}{\partial x} + \dfrac{1}{y} \cdot \dfrac{\partial z}{\partial y} = \dfrac{z}{y^2}$.

Dokazati.

Rj. $z = \dfrac{y}{f(\xi)}$ gdje je $\xi = x^2 - y^2$

$$\frac{\partial z}{\partial x} = \frac{0 \cdot f(\xi) - y \cdot \frac{\partial f}{\partial \xi} \cdot \frac{\partial \xi}{\partial x}}{f^2(\xi)} = \frac{-2xy \frac{\partial f}{\partial \xi}}{f^2(\xi)}$$

$$\frac{\partial z}{\partial y} = \frac{1 \cdot f(\xi) - y \cdot \frac{\partial f}{\partial \xi} \cdot \frac{\partial \xi}{\partial y}}{f^2(\xi)} = \frac{f(\xi) + 2y^2 \frac{\partial f}{\partial \xi}}{f^2(\xi)}$$

\# Ako je $x^2=v\cdot w$, $y^2=u\cdot w$, $z^2=u\cdot v$ i $f(x,y,z)=F(u,v,w)$ dokazati $x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}+z\cdot\frac{\partial f}{\partial z}=u\cdot\frac{\partial F}{\partial u}+v\cdot\frac{\partial F}{\partial v}+w\cdot\frac{\partial F}{\partial w}$.

Rj. $F(u,v,w)=f(x,y,z)=f(\sqrt{v\cdot w},\sqrt{u\cdot w},\sqrt{u\cdot v})$

$$\frac{\partial F}{\partial u}=\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$=f'_x\cdot 0+f'_y\cdot\frac{\sqrt{w}}{2\sqrt{u}}+f'_z\cdot\frac{\sqrt{v}}{2\sqrt{u}}=f'_y\cdot\frac{\sqrt{w}}{2\sqrt{u}}+f'_z\cdot\frac{\sqrt{v}}{2\sqrt{u}}$$

$$\frac{\partial F}{\partial v}=\frac{\partial f}{\partial x}\cdot\frac{\sqrt{w}}{2\sqrt{v}}+\frac{\partial f}{\partial y}\cdot 0+\frac{\partial f}{\partial z}\cdot\frac{\sqrt{u}}{2\sqrt{v}}=f'_x\cdot\frac{\sqrt{w}}{2\sqrt{v}}+f'_z\cdot\frac{\sqrt{u}}{2\sqrt{v}}$$

$$\frac{\partial F}{\partial w}=\frac{\partial f}{\partial x}\cdot\frac{\sqrt{v}}{2\sqrt{w}}+\frac{\partial f}{\partial y}\cdot\frac{\sqrt{u}}{2\sqrt{w}}+\frac{\partial f}{\partial z}\cdot 0=f'_x\cdot\frac{\sqrt{v}}{2\sqrt{w}}+f'_y\cdot\frac{\sqrt{u}}{2\sqrt{w}}$$

$$u\cdot\frac{\partial F}{\partial u}=\frac{\partial f}{\partial y}\cdot\frac{\sqrt{uw}}{2}+\frac{\partial f}{\partial z}\cdot\frac{\sqrt{uv}}{2}$$

$$v\cdot\frac{\partial F}{\partial v}=\frac{\partial f}{\partial x}\cdot\frac{\sqrt{vw}}{2}+\frac{\partial f}{\partial z}\cdot\frac{\sqrt{uv}}{2}$$

$$w\cdot\frac{\partial F}{\partial w}=\frac{\partial f}{\partial x}\cdot\frac{\sqrt{vw}}{2}+\frac{\partial f}{\partial y}\cdot\frac{\sqrt{uw}}{2}$$

Prema tome $x\cdot\frac{\partial f}{\partial x}+y\cdot\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=u\frac{\partial F}{\partial u}+v\frac{\partial F}{\partial v}+w\cdot\frac{\partial F}{\partial w}$

q.e.d.

\# ISPITNI ZADATAK

Ako je $z=z(x,y)$ i $x+y+z=f(x^2+y^2+z^2)$ provjeriti da li je tačna jednakost

$$(y-z)\cdot\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=x-y.$$

Rj. $z=z(x,y)\Rightarrow z$ je f-ja dvije promjenjive x i y.

$z=f(x^2+y^2+z^2)-x-y$

$t=x^2+y^2+z^2$

$s=-x-y$

$z=f(t)+s$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial x}+\frac{\partial s}{\partial x}$$

$$\frac{\partial z}{\partial x}=f'_t\cdot\left(2x+2z\frac{\partial z}{\partial x}\right)-1$$

$$\frac{\partial z}{\partial x}-f'_t\cdot 2z\frac{\partial z}{\partial x}=f'_t\cdot 2x-1$$

$$\frac{\partial z}{\partial x}=\frac{2xf'_t-1}{1-2zf'_t}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial y}+\frac{\partial s}{\partial y}$$

$$\frac{\partial z}{\partial y}=f'_t\cdot\left(2y+2z\frac{\partial z}{\partial x}\right)-1$$

$$\frac{\partial z}{\partial y}-2zf'_t\frac{\partial z}{\partial x}=2yf'_t-1$$

$$\frac{\partial z}{\partial y}=\frac{2yf'_t-1}{1-2zf'_t}$$

$$(y-z)\cdot\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=\frac{(y-z)(2xf'_t-1)}{1-2zf'_t}+\frac{(z-x)(2yf'_t-1)}{1-2zf'_t}=$$

$$=\frac{\cancel{2xyf'_t}-y-2xzf'_t\enclose{circle}{+z}+2yzf'_t\enclose{circle}{-z}\cancel{-2xyf'_t}+x}{1-2zf'_t}=$$

$$=\frac{(x-y)-2xzf'_t+2yzf'_t}{1-2zf'_t}=\frac{(x-y)+2zf'_t(-x+y)}{1-2zf'_t}=$$

$$=\frac{(x-y)(1-2zf'_t)}{1-2zf'_t}=x-y$$

(#) Ako je $z=\frac{y}{f(x^2-y^2)}$, gdje je f diferencijabilna f-ja, izračunati $\frac{1}{x}\cdot\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}$.

Rj. $z=y\,f^{-1}(x^2-y^2)=y\,f^{-1}(u)$, gdje je $u=x^2-y^2$

$$\frac{\partial z}{\partial x}=y(-1)f_u^{-2}(x^2-y^2)\cdot 2x=\frac{-2xy}{f_u^2(x^2+y^2)}$$

$$\frac{\partial z}{\partial y}=\left(y\,f^{-1}(u)\right)_y'=1\cdot f^{-1}(u)+y\cdot(-1)f_u^{-2}(u)\cdot(-2y)=$$

$$=\frac{1}{f(x^2-y^2)}+\frac{2y^2}{f_u^2(x^2+y^2)}$$

$$\frac{1}{x}\cdot\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}=\frac{-2y}{f_u^2(x^2+y^2)}+\frac{1}{y\,f(x^2-y^2)}+\frac{2y}{f_u^2(x^2+y^2)}=$$

$$=\frac{1}{y\,f(x^2-y^2)}=\frac{1}{y^2}\cdot\frac{y}{f(x^2-y^2)}=\frac{z}{y^2}$$

prema tome $\frac{1}{x}\cdot\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}=\frac{z}{y^2}$.

(#) Ako je $z=e^y\varphi(y\,e^{\frac{x^2}{2y^2}})$ gdje je φ diferencijabilna f-ja, dokazati da je $(x^2-y^2)\frac{\partial z}{\partial x}+xy\frac{\partial z}{\partial y}=xyz$.

Rj. $z=e^y\varphi(\xi)$, gdje je $\xi(x,y)=y\,e^{\frac{x^2}{2y^2}}$

$$\frac{\partial \xi}{\partial x}=y\,e^{\frac{x^2}{2y^2}}\cdot 2\cdot\frac{x}{2y^2}=\frac{x}{y}\,e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial \xi}{\partial y}=e^{\frac{x^2}{2y^2}}+y\,e^{\frac{x^2}{2y^2}}\left(\frac{1}{2}x^2y^{-2}\right)_y'=e^{\frac{x^2}{2y^2}}+y\,e^{\frac{x^2}{2y^2}}\left(\frac{1}{2}x^2\cdot(-2)y^{-3}\right)$$

$$=e^{\frac{x^2}{2y^2}}-\frac{x^2}{y^2}\,e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial z}{\partial x}=e^y\cdot\frac{\partial\varphi}{\partial\xi}\cdot\frac{\partial\varphi}{\partial x}=\frac{x}{y}\,e^y e^{\frac{x^2}{2y^2}}\,\frac{\partial\varphi}{\partial\xi}$$

$$\frac{\partial z}{\partial y}=e^y\varphi(\xi)+e^y\cdot\frac{\partial\varphi}{\partial\xi}\cdot\frac{\partial\xi}{\partial y}=e^y\varphi(\xi)+e^ye^{\frac{x^2}{2y^2}}\cdot\frac{\partial\varphi}{\partial\xi}-$$

$$-e^y\cdot\frac{x^2}{y^2}\,e^{\frac{x^2}{2y^2}}\,\frac{\partial\varphi}{\partial\xi}$$

$$(x^2-y^2)\frac{\partial z}{\partial x}+xy\frac{\partial z}{\partial y}=(x^2-y^2)\cdot\frac{x}{y}\,e^{y+\frac{x^2}{2y^2}}\frac{\partial\varphi}{\partial\xi}+$$

$$+xy\left(e^y\varphi(\xi)+e^{y+\frac{x^2}{2y^2}}\cdot\frac{\partial\varphi}{\partial\xi}-\frac{x^2}{y^2}\,e^{y+\frac{x^2}{2y^2}}\cdot\frac{\partial\varphi}{\partial\xi}\right)$$

$$=\underline{\frac{x^3}{y}\,e^{y+\frac{x^2}{2y^2}}\frac{\partial\varphi}{\partial\xi}}\;\underline{\underline{-yx\,e^{y+\frac{x^2}{2y^2}}\frac{\partial\varphi}{\partial\xi}}}+xy\,e^y\varphi(\xi)+$$

$$\underline{\underline{+xy\,e^{y+\frac{x^2}{2y^2}}\frac{\partial\varphi}{\partial\xi}}}\;\underline{-\frac{x^3}{y}\,e^{y+\frac{x^2}{2y^2}}\cdot\frac{\partial\varphi}{\partial\xi}}=$$

$$=xy\,e^y\varphi(\xi)=xy\,e^y\varphi(y\,e^{\frac{x^2}{2y^2}})=xyz$$

Parcijalni izvodi i diferencijali višeg reda f-ja dvije i više promjenjivih

Parcijalnim izvodima drugog reda f-je $z=f(x,y)$ nazivamo parcijalnim izvodima njenih parcijalnih izvoda prvog reda. Za parcijalne izvode drugog reda upotrebljavamo ove oznake

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x^2}=f''_{xx}(x,y) \qquad \boxed{\begin{matrix}\partial\\ \text{DELTA}\end{matrix}}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial^2 z}{\partial x\partial y}=f''_{xy}(x,y) \quad \text{itd.}$$

Analogno se definiraju i označavaju izvodi viših redova.

Diferencijalom drugog reda f-je $z=f(x,y)$ nazivamo diferencijal diferencijala prvog reda te f-je za fiksirane priraste nezavisnih varijabli.

$$d^2z=d(dz)$$

Analogno se određuju diferencijali f-je z višega nego drugog reda, na primjer $d^3z=d(d^2z)$

i općenito $d^n z=d(d^{n-1}z)$ $(n=2,3,\dots)$

Ako je $z=f(x,y)$ gdje su x i y nezavisne varijable i f-ja f ima neprekidne parcijalne izvode drugog reda, tada se diferencijal drugog reda f-je z računa po formuli

$$d^2z=\frac{\partial^2 z}{\partial x^2}dx^2+2\frac{\partial^2 z}{\partial x\partial y}dxdy+\frac{\partial^2 z}{\partial y^2}dy^2.$$

Općenito, kada postoje neprekidne odgovarajuće derivacije, vrijedi simbolička formula

$$d^n z=\left(dx\frac{\partial}{\partial x}+dy\frac{\partial}{\partial y}\right)^n z,$$

koja se formalno razvije po binomnom zakonu.

Ⓗ Nađi parcijalne izvode drugog reda f-je

a) $z=e^{-xy}$ c) $u=x^3y+y^3x+z^3y$ e)✓ $z=\ln tg\frac{x}{y}$

b) $z=x^3+y^3-xy$ d)✓ $u=\ln(x+y-z)$ f)✓ $u=\sin(x^2+y+z^3)$

Rj. a) $z=e^{-xy}$

$$\frac{\partial z}{\partial x}=e^{-xy}\cdot(-y)=-ye^{-xy} \qquad \frac{\partial^2 z}{\partial x^2}=(-y)e^{-xy}\cdot(-y)=y^2e^{-xy}$$

$$\frac{\partial z}{\partial y}=e^{-xy}\cdot(-x)=-xe^{-xy} \qquad \frac{\partial^2 z}{\partial y^2}=(-x)e^{-xy}\cdot(-x)=x^2e^{-xy}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=-e^{-xy}-ye^{-xy}(-x)=$$
$$=e^{-xy}(xy-1)$$

b) $z=x^3+y^3-xy$

$$\frac{\partial z}{\partial x}=3x^2-y \qquad \frac{\partial^2 z}{\partial x^2}=6x, \quad \frac{\partial^2 z}{\partial y^2}=6y, \quad \frac{\partial^2 z}{\partial x\partial y}=-1$$

$$\frac{\partial z}{\partial y}=3y^2-x$$

c) $u=x^3y+y^3x+z^2y$

$$\frac{\partial u}{\partial x}=3x^2y+y^3 \qquad \frac{\partial^2 u}{\partial x^2}=6xy, \quad \frac{\partial^2 u}{\partial y^2}=6xy, \quad \frac{\partial^2 u}{\partial z^2}=6yz$$

$$\frac{\partial u}{\partial y}=x^3+3y^2x+z^3 \qquad \frac{\partial^2 u}{\partial x\partial y}=3x^2+3y^2, \quad \frac{\partial^2 u}{\partial x\partial z}=0$$

$$\frac{\partial u}{\partial z}=3z^2y \qquad \frac{\partial^2 u}{\partial y\partial z}=3z^2$$

d) $u=\ln(x+y-z)$

$$\frac{\partial u}{\partial x}=\frac{1}{x+y-z} \qquad \frac{\partial u}{\partial z}=\frac{-1}{x+y-z} \qquad \frac{\partial^2 u}{\partial y^2}=\frac{-1}{(x+y-z)^2}$$

$$\frac{\partial u}{\partial y}=\frac{1}{x+y-z} \qquad \frac{\partial^2 u}{\partial x^2}=\frac{-1}{(x+y-z)^2} \qquad \frac{\partial^2 u}{\partial z^2}=\frac{-1}{(x+y-z)^2}$$

završiti sami

…

Provjeriti da li vrijedi

a) $u = \ln(x^2+y^2) \implies \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

b) $u = e^{-\alpha x}\cdot\varphi(x-y) \implies \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\alpha\cdot\frac{\partial u}{\partial y} = \alpha^2 u$

Rj. a) $\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2}$ $\qquad \frac{\partial^2 u}{\partial x^2} = \frac{2(x^2+y^2)-2x\cdot(2x)}{(x^2+y^2)^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2}$

$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2}$ $\qquad \frac{\partial^2 u}{\partial y^2} = \frac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2} = \frac{2x^2-2y^2}{(x^2+y^2)^2}$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{2y^2-2x^2}{(x^2+y^2)^2} + \frac{2x^2-2y^2}{(x^2+y^2)^2} = \frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2} = 0$$ što je i trebalo dobiti

b) $u = e^{-\alpha x}\cdot\varphi(x-y)$

$$\frac{\partial u}{\partial x} = e^{-\alpha x}\cdot(-\alpha)\,\varphi(x-y) + e^{-\alpha x}\cdot\varphi'_x = e^{-\alpha x}\left[-\alpha\varphi(x-y)+\varphi'_x\right]$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-\alpha x}\cdot(-\alpha)\left(-\alpha\varphi(x-y)+\varphi'_x\right) + e^{-\alpha x}\left[-\alpha\varphi'_x+\varphi''_{xx}\right]$$

$$= e^{-\alpha x}\left(\alpha^2\varphi(x-y)-\alpha\varphi'_x-\alpha\varphi'_x+\varphi''_{xx}\right) = e^{-\alpha x}\left(\alpha^2\varphi(x-y)-2\alpha\varphi'_x+\varphi''_{xx}\right)$$

$$\frac{\partial u}{\partial y} = e^{-\alpha x}\cdot\varphi'_y\cdot(-1) = -e^{\alpha x}\varphi'_y$$

$$\frac{\partial^2 u}{\partial y^2} = -e^{\alpha x}\varphi''_{yy}\cdot(-1) = e^{\alpha x}\cdot\varphi''_{yy}$$

$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} - 2\alpha\frac{\partial u}{\partial y} = e^{-\alpha x}\left(\alpha^2\varphi(x-y)-2\alpha\varphi'_x+\varphi''_{xx}-\varphi''_{yy}+2\alpha\varphi'_y\right) =$ (u slučaju da je $\varphi'_x = \varphi'_y$ i $\varphi''_{xx} = \varphi''_{yy}$) $= \alpha^2 e^{-\alpha x}\varphi(x-y) = \alpha^2 u$

Naći parcijalne izvode prvog i drugog reda f-je $z = \ln(x^2+y^2)$.

Rj.

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2}(x^2+y^2)'_x = \frac{2x}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2}(x^2+y^2)'_y = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{2x}{x^2+y^2}\right)'_x = 2\left(\frac{x}{x^2+y^2}\right)'_x = 2\,\frac{1\cdot(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}$$

$$= 2\cdot\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = \left(\frac{2x}{x^2+y^2}\right)'_y = 2\,\frac{1\cdot(x^2+y^2)-x\cdot 2y}{(x^2+y^2)^2} = 2\cdot\frac{x^2-2xy+y^2}{(x^2+y^2)^2}$$

$$= 2\cdot\frac{(x-y)^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(\frac{2y}{x^2+y^2}\right)'_y = 2\left(\frac{y}{x^2+y^2}\right)'_y = 2\,\frac{1\cdot(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2}$$

$$= 2\,\frac{x^2-y^2}{(x^2+y^2)^2}$$

Parcijalni izvodi višeg reda složenih f-ja

(#) Ako je $u=\varphi(\xi,\eta)$ pričemu je $\xi = x+y$, $\eta = x-y$ izračunati izvode $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x\partial y}$, $\frac{\partial^2 u}{\partial y^2}$.

Rj.

$$\frac{\partial u}{\partial x} = \frac{\partial\varphi}{\partial\xi}\cdot\frac{\partial\xi}{\partial x} + \frac{\partial\varphi}{\partial\eta}\cdot\frac{\partial\eta}{\partial x} = \frac{\partial\varphi}{\partial\xi}+\frac{\partial\varphi}{\partial\eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial\xi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial\eta}\right) = \frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial\xi}{\partial x} + \frac{\partial^2\varphi}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial x} +$$

$$+ \frac{\partial^2\varphi}{\partial\eta\partial\xi}\cdot\frac{\partial\xi}{\partial x} + \frac{\partial^2\varphi}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial x} = \frac{\partial^2\varphi}{\partial\xi^2} + 2\frac{\partial^2\varphi}{\partial\xi\partial\eta} + \frac{\partial^2\varphi}{\partial\eta^2}$$

$$= \left(\frac{\partial\varphi}{\partial\xi}+\frac{\partial\varphi}{\partial\eta}\right)^2$$

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial\xi}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial\eta}\right) = \frac{\partial^2\varphi}{\partial\xi^2}\frac{\partial\xi}{\partial y}+\frac{\partial^2\varphi}{\partial\xi\partial\eta}\cdot\frac{\partial\eta}{\partial y} +$$

$$+ \frac{\partial^2\varphi}{\partial\eta\partial\xi}\cdot\frac{\partial\xi}{\partial y}+\frac{\partial^2\varphi}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial y} = \frac{\partial^2\varphi}{\partial\xi^2} - \frac{\partial^2\varphi}{\partial\eta^2}$$

(#) Ako je $u = \frac{\varphi(x-y)+\psi(x+y)}{x}$, gdje su φ i ψ diferencijabilne f-je izračunati $\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right) - x^2\frac{\partial^2 u}{\partial y^2}$.

Rj.

$$u = \frac{1}{x}(\varphi(x-y)+\psi(x+y)) = x^{-1}(\varphi(x-y)+\psi(x+y))$$

$$u'_x = \frac{\partial u}{\partial x} = (-1)x^{-2}(\varphi(\underbrace{x-y}_{s})+\psi(\underbrace{x+y}_{t})) + \frac{1}{x}(\varphi'_s\cdot s'_x + \psi'_t\cdot t'_x) =$$

$$= \frac{-1}{x^2}[\varphi(x-y)+\psi(x+y)] + \frac{1}{x}(\varphi'_s\cdot 1 + \psi'_t\cdot 1)$$

$$x^2\frac{\partial u}{\partial x} = -\varphi(x-y) - \psi(x+y) + x(\varphi'_s+\psi'_t)$$ gdje su $s=x-y$ i $t=x+y$

$$\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right) = \underline{-\varphi'_s\cdot 1} - \underline{\underline{\psi'_t\cdot 1}} + \underline{1\cdot(\varphi'_s} + \underline{\underline{\psi'_t}}) + x(\varphi''_{ss}\cdot 1 + \psi''_{tt}\cdot 1)$$

$$= x(\varphi''_{ss}+\psi''_{tt}) \qquad \ldots(1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x}(\varphi'_s\cdot s'_y + \psi'_t\cdot t'_y) = \frac{1}{x}(\varphi'_s\cdot(-1) + \psi'_t\cdot 1) = \frac{1}{x}(-\varphi'_s+\psi'_t)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{x}(-\varphi''_{ss}\cdot s'_y + \psi''_{tt}\cdot t'_y) = \frac{1}{x}(\varphi''_{ss}+\psi''_{tt})$$

$$x^2\frac{\partial^2 u}{\partial y^2} = x(\varphi''_{ss}+\psi''_{tt}) \qquad \ldots(2)$$

$$\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right) - x^2\frac{\partial^2 u}{\partial y^2} \overset{(1)i(2)}{=\!=} 0$$ traženo rješenje

Zadaci za vježbu

§ 3. Izvodi i diferencijali funkcija više promenljivih

Parcijalni izvodi

3032. Zapremina gasa v je funkcija njegove temperature i pritiska: $v=f(p, T)$. Kad pritisak gasa ostaje konstantan, srednjim koeficijentom širenja gasa pri promeni njegove temperature od T_1 do T_2 naziva se veličina $\frac{v_2-v_1}{v(T_2-T_1)}$. Šta treba zvati koeficijentom širenja gasa pri konstantnom pritisku za datu temperaturu T_0?

3033. Temperatura θ u datoj tački A štapa Ox je funkcija apscise x tačke A i vremena t: $0=f(x, t)$. Kakav fizički smisao imaju parcijalni izvodi $\frac{\partial\theta}{\partial t}$ i $\frac{\partial\theta}{\partial x}$?

3034. Površina S pravougaonika čija je osnovica b i visina h izražava se obrascem $S=bh$. Naći $\frac{\partial S}{\partial h}$ i $\frac{\partial S}{\partial x}$ i objasniti geometrijski smisao rezultata.

3035. Date su dve funkcije: $u=\sqrt{a^2-x^2}$ (a je konstanta) i $z=\sqrt{y^2-x^2}$. Naći $\frac{du}{dx}$ i $\frac{\partial z}{\partial x}$ i uporediti rezultate.

U zadacima 3036—3084 naći parcijalne izvode datih funkcija po svakoj od nezavisno promenljivih (x, y, z, u, v, t, φ i ψ su promenljive veličine).

3036. $z=x-y$.

3037. $z=x^3y-y^3x$.

3038. $\theta=axe^{-t}+bt$ (a, b su konstante).

3039. $z=\frac{u}{v}+\frac{v}{u}$.

3040. $z=\frac{x^3+y^3}{x^2+y^2}$.

3041. $z=(5x^2y-y^3+7)^3$.

3042. $z=x\sqrt{y}+\frac{y}{\sqrt[3]{x}}$.

3043. $z=\ln(x+\sqrt{x^2+y^2})$.

3044. $z=\operatorname{arctg}\frac{x}{y}$.

3045. $z=\frac{1}{\operatorname{arctg}\frac{y}{x}}$.

3046. $z=x^y$.

3047. $z=\ln(x^2+y^2)$.

3048. $z=\ln\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}$.

3049. $z=\arcsin\frac{\sqrt{x^2-y^2}}{\sqrt{x^2+y^2}}$.

3050. $z=\ln\operatorname{tg}\frac{x}{y}$.

3051. $z=e^{-\frac{x}{y}}$.

3052. $z=\ln(x+\ln y)$.

3053. $u=\operatorname{arctg}\frac{v+w}{v-w}$.

3054. $z=\sin\frac{x}{y}\cos\frac{y}{x}$.

3055. $z=\left(\frac{1}{3}\right)^{\frac{y}{x}}$.

3056. $z=(1+xy)^y$.

3057. $z=xy\ln(x+y)$.

3058. $z=x^{xy}$.

3059. $u=xyz$.

3060. $u=xy+yz+zx$.

3061. $u=\sqrt{x^2+y^2+z^2}$.

3062. $u=x^3+yz^2+3yx-x+z$.

3063. $w=xyz+yzv+zvk+vxy$.

3064. $u=e^{x(x^2+y^2+z^2)}$.

3066. $u=\ln(x+y+z)$

3065. $u=\sin(x^2+y^2+z^2)$.

3075. $z=\operatorname{arctg}\sqrt{x^y}$.

3067. $u=x^{\frac{y}{z}}$.

3068. $u=x^{yz}$.

3069. $f(x, y)=x+y-\sqrt{x^2+y^2}$ u tački (3, 4).

3070. $z=\ln\left(x+\frac{y}{2x}\right)$ u tački (1, 2).

3071. $z=(2x+y)^{2x+y}$.

3072. $z=(1+\log_y x)^3$.

3073. $z=xye^{\sin\pi xy}$.

3074. $z=(x^2+y^2)\frac{1-\sqrt{x^2+y^2}}{1+\sqrt{x^2-y^2}}$.

3076. $z=2\sqrt{\frac{1-\sqrt{xy}}{1+\sqrt{xy}}}$.

3077. $z=\ln[xy^2+yx^2+\sqrt{1+(xy^2+yx^2)^2}]$.

3078. $z=\sqrt{1-\left(\frac{x+y}{xy}\right)^2}+\arcsin\frac{x+y}{xy}$.

3079. $z=\operatorname{arctg}\left(\operatorname{arctg}\frac{y}{x}\right)-\frac{1}{2}\frac{\operatorname{arctg}\frac{x}{y}-1}{\operatorname{arctg}\frac{x}{y}+1}-\operatorname{arctg}\frac{x}{y}$.

3080. $u=\frac{k}{(x^2+y^2+z^2)^2}$.

3081. $u=\operatorname{arctg}(x-y)^z$.

3082. $u=(\sin x)^{yz}$.

3083. $u=\ln\frac{1-\sqrt{x^2+y^2+z^2}}{1+\sqrt{x^2+y^2+z^2}}$.

3084. $w=\frac{1}{2}\operatorname{tg}^2(x^2y^2+z^2v^2-xyzv)+\ln\cos(x^2y^2+z^2v^2-xyzv)$.

3085. $n=\frac{\cos(\varphi-2\psi)}{\operatorname{sos}(\varphi+2\psi)}$. Naći $\left(\frac{\partial u}{\partial\psi}\right)_{\substack{\varphi=\frac{\pi}{4}\\ \psi=\pi}}$

3086. $u=\sqrt{az^3-bt^3}$. Naći $\frac{\partial u}{\partial z}$ i $\frac{\partial u}{\partial t}$ za $z=b$, $t=a$.

3087. $z=\frac{x\cos y-y\cos x}{1+\sin x+\sin y}$. Naći $\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$ za $x=y=0$.

3088. $u=\sqrt{\sin^2 x+\sin^2 y+\sin^2 z}$. Naći $\left(\frac{\partial u}{\partial z}\right)_{\substack{x=0\\ y=0\\ z=\frac{\pi}{4}}}$

3089. $u=\ln(1+x+y^2+z^3)$. Naći $u'_x+u'_y+u'_z$ za $x=y=z=1$.

3090. $f(x, y)=x^3y-y^3x$. Naći $\left(\frac{\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial y}}\right)_{\substack{x=1\\ y=2}}$

3091. Koliki ugao zaklapa tangenta u tački (2, 4, 5) krive $\begin{cases} z=\dfrac{x^2+y^2}{4} \\ y=4 \end{cases}$ sa pozitivnim pravcem apscisne ose.

3092. Koliki ugao zaklapa tangenta krive $\begin{cases} z=\sqrt{1+x^2+y^2} \\ x=1 \end{cases}$ u tački $(1, 1, \sqrt{3})$ sa pozitivnim pravcem ordinatne ose.

3093. Pod kojim se uglom seku ravne krive po kojima ravan $y=2$ preseca površine $z=x^2+\frac{y^2}{6}$ i $z=\frac{x^2+y^2}{3}$?

Diferencijali. Približna računanja

U zadacima 3094—3097 naći parcijalne diferencijale datih funkcija po svakoj od nezavisno promenljivih.

3094. $z=xy^3-3x^2y^2+2y^4$.

3095. $z=\sqrt{x^2+y^2}$.

3096. $z=\frac{xy}{x^2+y^2}$.

3097. $u=\ln(x^3+2y^3-z^3)$.

3098. $z=\sqrt[3]{x+y^2}$. Naći $d_y z$ za $x=2$, $y=5$, $\Delta y=0{,}01$.

3099. $z=\sqrt{\ln xy}$. Naći $d_x z$ za $x=1$, $y=1$, 2, $\Delta x=0{,}016$.

3100. $u=p\frac{qr}{p}+\sqrt{p+q+r}$. Naći $d_p u$ za $p=1$, $q=3$, $r=5$, $\Delta p=0{,}01$.

U zadacima 3101—3109 naći totalne diferencijale datih funkcija

3101. $z=x^2y^4-x^3y^3+x^4y^3$.

3102. $z=\frac{1}{2}\ln(x^2+y^2)$.

3103. $z=\frac{x+y}{x-y}$.

3104. $z=\arcsin\frac{x}{y}$.

3105. $z=\sin(xy)$.

3106. $z=\operatorname{arctg}\frac{x+y}{1-xy}$.

3107. $z=\frac{x^2+y^2}{x^2-y^2}$.

3108. $z=\operatorname{arctg}(xy)$.

3109. $u=x^{yz}$.

§ 4. Diferenciranje funkcija

Posredna funkcija

3124. $u=e^{x-2y}$, pri čemu je $x=\sin t$, $y=t^3$; $\frac{du}{dt}=?$

3125. $u=z^2+y^2+zy$, $z=\sin t$, $y=e^t$; $\frac{du}{dt}=?$

3126. $z=\arcsin(x-y)$, $x=3t$, $y=4t^3$; $\frac{dz}{dt}=?$

3127. $z=x^2y-y^2x$, gde je $x=u\cos v$, $y=u\sin v$; $\frac{\partial z}{\partial u}=?$ $\frac{\partial z}{\partial v}=?$

3128. $z=x^2\ln y$, $x=\frac{u}{v}$, $y=3u-2v$ $\frac{\partial z}{du}=?$ $\frac{\partial z}{\partial v}=?$

3129. $u=\ln(e^x-e^y)$; $\frac{\partial u}{\partial x}=?$ Naći $\frac{du}{dx}$, Ako je $y=x^3$.

3130. $z=\operatorname{arctg}(xy)$; naći $\frac{dz}{dx}$, ako je $y=e^x$.

3131. $u=\arcsin\frac{x}{z}$, gde je $z=\sqrt{x^2+1}$; $\frac{du}{dx}=?$

3132. $z=\operatorname{tg}(3t+2x^2-y)$, $x=\frac{1}{t}$, $y=\sqrt{t}$; $\frac{dz}{dt}=?$

3133. $u=\frac{e^{ax}(x-z)}{a^2+1}$, $y=a\sin x$, $z=\cos x$; $\frac{du}{dx}=?$

3134. $z=\frac{xy\operatorname{arctg}(xy+x+y)}{x+y}$; $dz=?$

3135. $z=(x^2+y^2)e^{\frac{x^2+y^2}{xy}}$; $\frac{\partial z}{\partial x}=?$ $\frac{\partial z}{\partial y}=?$ $dz=?$

3136. $z=f(x^2-y^2, e^{xy})$; $\frac{\partial z}{\partial x}=?$ $\frac{\partial z}{\partial y}=?$

3137. Uveriti se da funkcija $z=\operatorname{arctg}\frac{x}{y}$, u kojoj je $x=u+v$, $y=u-v$, zadovoljava relaciju

$$\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}=\frac{u-v}{v^2+u^2}.$$

3138. Uveriti se da funkcija $z=\varphi(x^2+y^2)$, u kojoj je φ diferencijabilna funkcija, zadovoljava relaciju:

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0.$$

3139. $u=\sin x+F(\sin y-\sin x)$; uveriti se da je $\frac{\partial u}{\partial y}\cos x+\frac{\partial u}{\partial x}\cos y=\cos x\cos y$, ma kakva bila diferencijabilna funkcija F.

3140. $z=\frac{y}{f(x^2-y^2)}$, uveriti se da je $\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{y}{y^2}$, ma kakva bila diferencijabilna funkcija f.

3141. Pokazati da homogena diferencijabilna funkcija $z=F\left(\frac{y}{x}\right)$ nultog stepena homogenosti (vidi zad. 2961) zadovoljava relaciju $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$.

3142. Pokazati da homogena funkcija $u=x^k F\left(\frac{z}{x};\ \frac{y}{x}\right)$, k-tog stepena homogenosti, u kojoj je F diferencijabilna funkcija, zadovoljava relaciju

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=ku.$$

3143. Proveriti tvrđenje formulisano u zadatku 3142 na funkciji

$$u=x^5\sin\frac{z^2+y^2}{x^2}.$$

3144. Neka je funkcija $f(x, y)$ diferencijabilna. Dokazati da, ako se promenljive x i y zamene linearnim homogenim funkcijama promenljivih X i Y, onda je tako dobijena funkcija $F(X, Y)$ vezana sa funkcijom $f(x, y)$ sledećom relacijom:

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=X\frac{\partial F}{\partial X}+Y\frac{\partial F}{\partial Y}.$$

§ 5. Izvodi višeg reda

3181. $z=x^3+xy^2-5xy^3+y^5$. Uveriti se da je: $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial y\,\partial x}$.

3182. $z=x^y$. Uveriti se da je $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial y\,\partial x}$.

3183. $z=e^x(\cos y+x\sin y)$. Uveriti se da je

$$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial y\,\partial x}.$$

3184. $z=\operatorname{arctg}\frac{y}{x}$. Uveriti se da je $\frac{\partial^3 z}{\partial y^2\,\partial x}=\frac{\partial^3 z}{\partial x\,\partial y^2}$.

U zadacima 3185—3192 naći $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\,\partial y}$, i $\frac{\partial^2 z}{\partial y^2}$ za date frnkcije.

3185. $z=\frac{1}{3}\sqrt{(x^2+y^2)^3}$.

3186. $z=\ln(x+\sqrt{x^2+y^2})$.

3187. $z=\operatorname{arctg}\frac{x+y}{1-xy}$.

3188. $z=\sin^2(ax+by)$.

3189. $z=e^{xe^y}$.

3190. $z=\frac{x-y}{x+y}$.

3191. $z=y^{\ln x}$.

3192. $z=\arcsin(xy)$.

3193. $u=\sqrt{x^2+y^2+z^2-2xz}$; $\frac{\partial^2 u}{\partial y\,\partial z}=?$

3194. $z=e^{xy^2}$; $\frac{\partial^3 z}{\partial x^2\,\partial y}=?$

3195. $s=\ln(x^2+y^2)$; $\frac{\partial^3 z}{\partial x\,\partial y^2}=?$

3196. $z=\sin xy$; $\frac{\partial^3 z}{\partial x\,\partial y^2}=?$

3197. $w=e^{xyz}$; $\frac{\partial^3 w}{\partial x\,\partial y\,\partial z}=?$

3198. $v=x^m y^n z^p$; $\frac{\partial^6 v}{\partial x\,\partial y^3\,\partial z^2}=?$

3199. $z=\ln(e^x+e^y)$; uveriti se da je $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$ i da je

$$\frac{\partial^2 z}{\partial x^2}\,\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2=0.$$

3200. $u=e^x(x\cos y-y\sin y)$. Pokazati da je $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

3201. $u=\ln\frac{1}{\sqrt{x^2+y^2}}$; pokazati da je $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

3202. $u=\frac{1}{\sqrt{x^2+y^2+z^2}}$; pokazati da je $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$.

3203. $r=\sqrt{x^2+y^2+z^2}$; pokazati da je

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}=\frac{2}{r},\quad \frac{\partial\,(\ln r)}{\partial x^2}+\frac{\partial^2(\ln r)}{\partial y^2}+\frac{\partial^2(\ln r)}{\partial z^2}=\frac{1}{r^2}.$$

3204. Za koje vrednosti konstante a funkcija $v=x^3+axy^2$ zadovoljava jednačinu

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0?$$

3205. $z=\frac{y}{y^2-a^2x^2}$; pokazati da je $\frac{\partial^2 z}{\partial x^2}=a^2\frac{\partial^2 z}{\partial y^2}$.

3206. $v=\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}$; uveriti se da je

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+2\left(\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial y\,\partial z}+\frac{\partial^2 v}{\partial z\,\partial x}\right)=0.$$

3207. $z=f(x, y)$, $\xi=x+y$, $\eta=x-y$; uveriti se da je

$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=4\frac{\partial^2 z}{\partial\xi\,\partial\eta}.$$

3208. $v=x\ln(x+r)-r$, gde je $r^2=x^2+y^2$. Uveriti se da je

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{1}{x+r}.$$

3209. Izvesti obrazac za drugi izvod $\frac{d^2 y}{dx^2}$ funkcije y, definisane implicitno jednačinom $f(x, y)=0$.

3210. $y=\varphi(x-at)+\psi(x+at)$. Pokazati da je

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2},$$

ma kakve bile dvaput diferencijabilne funkcije φ i ψ.

3211. $u=\varphi(x)+\psi(y)+(x-y)\psi'(y)$. Uveriti se da je

$$(x-y)\frac{\partial^2 y}{\partial x\,\partial y}=\frac{\partial u}{\partial y}$$

(φ i ψ su dvaput diferencijabilne funkcije).

3212. $z=y\varphi(x^2-y^2)$. Uveriti se da je

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}$$

(φ je diferencijabilna funkcija).

3213. $r=x\varphi(x+y)+y\psi(x+y)$; pokazati da je

$$\frac{\partial^2 r}{\partial x^2}-2\frac{\partial^2 r}{\partial x\partial y}+\frac{\partial^2 r}{\partial y^2}=0$$

(φ i ψ su dvaput diferencijabilne funkcije).

3214. $u=\frac{1}{y}[\varphi(ax+y)+\psi(ax-y)]$. Pokazaii da je

$$\frac{\partial^2 u}{\partial x^2}=\frac{a^2}{y^2}\cdot\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right).$$

3215. $u=\frac{1}{x}[\varphi(x-y)+\psi(x+y)]$. Pokazati da je

$$\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)=x^2\frac{\partial^2 u}{\partial y^2}.$$

3216. $u=xe^y+ye^x$. Pokazati da je

$$\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 y}{\partial y^3}=x\frac{\partial^3 u}{\partial x\partial y^2}+y\frac{\partial^3 u}{\partial x^2\partial y}.$$

3217. $u=e^{xyz}$. Pokazati da je

$$\frac{\partial^3 y}{\partial x\partial y\partial z}=xy\frac{\partial^2 u}{\partial x\partial y}+2x\frac{\partial u}{\partial x}+u.$$

3218. $u=\ln\frac{x^2-y^2}{xy}$. Pokazati da je

$$\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial x^2\partial y}-\frac{\partial^3 u}{\partial x\partial y^2}-\frac{\partial^3 u}{\partial y^3}=2\left(\frac{1}{y^3}-\frac{1}{x^3}\right).$$

U zadacima 3219—3224 naći diferencijale drugog reda za đate funkcije.

3219. $z=xy^2-x^2y$. **3220.** $z=\ln(x-y)$.

3221. $z=\frac{1}{2(x^2+y^2)}$. **3222.** $z=x\sin^2 y$.

3223. $z=e^{xy}$. **3224.** $u=xyz$.

3225. $z=\sin(2x+y)$. Naći d^3z u tačkama $(0, \pi)$; $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

3226. $u+\sin(x+y+z)$; $d^2u=?$

3227. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; $d^2z=?$

3228. $z^3-3xyz=a^3$; $d^2z=?$

3229. $3x^2y^2+2z^2xy-2zx^3+4zy^3-4=0$. Naći d^2z u tački $(2, 1, 2)$.

Rješenja

3032. $\frac{1}{v}\frac{\partial v}{\partial T}$ za $T=T_0$.

3033. $\frac{\partial\theta}{\partial t}$ — brzina menjanja temperature u datoj tački; $\frac{\partial\theta}{\partial x}$ — brzina menjanja temperature u odnosu na dužinu (duž štapa), u datom trenutku vremena.

3034. $\frac{\partial S}{\partial h}=b$ — brzina menjanja površine u zavisnosti od visine pravougaonika; $\frac{\partial S}{\partial h}=h$ — brzina menjanja površine u zavisnosti od osnovice pravougaonika.

3036. $\frac{\partial z}{\partial x}=1$, $\frac{\partial z}{\partial y}=-1$. **3037.** $\frac{\partial z}{\partial x}=3x^2y-y^3$; $\frac{\partial z}{\partial y}=x^3-3y^2x$.

3038. $\frac{\partial\theta}{\partial x}=ae^{-t}$; $\frac{\partial\theta}{\partial t}=-axe^{-t}+b$.

3039. $\frac{\partial z}{\partial u}=\frac{1}{v}-\frac{v}{u^2}$; $\frac{\partial z}{\partial v}=-\frac{u}{v^2}+\frac{1}{u}$.

3040. $\frac{\partial z}{\partial x}=\frac{x^4+3x^2y^2-2xy^3}{(x^2+y^2)^2}$; $\frac{\partial z}{\partial y}=\frac{y^4+3x^2y^2-2x^3y}{(x^2+y^2)^2}$.

3041. $\frac{\partial z}{\partial x}=30xy(5x^2y-y^3+7)^2$; $\frac{\partial z}{\partial y}=3(5x^2y-y^3+7)^2(5x^2-3y^2)$.

3042. $\frac{\partial z}{\partial x}=\sqrt{y}-\frac{y}{3\sqrt[3]{x^4}}$; $\frac{\partial z}{\partial y}=\frac{x}{2\sqrt{y}}+\frac{1}{\sqrt[3]{x}}$.

3043. $\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}$; $\frac{\partial z}{\partial y}=\frac{y}{x^2+y^2+x\sqrt{x^2+y^2}}$.

3044. $\frac{\partial z}{\partial x}=\frac{y}{x^2+y^2}$; $\frac{\partial z}{\partial y}=-\frac{x}{x^2+y^2}$.

3045. $\frac{\partial z}{\partial x}=\frac{y}{(x^2+y^2)\left(\operatorname{arctg}\frac{y}{x}\right)^2}$; $\frac{\partial z}{\partial y}=-\frac{x}{(x^2+y^2)\left(\operatorname{arctg}\frac{y}{x}\right)^2}$.

3046. $\frac{\partial z}{\partial x}=yx^{y-1}$; $\frac{\partial z}{\partial y}=x^y\ln x$. **3047.** $\frac{\partial z}{\partial x}=\frac{2x}{x^2+y^2}$; $\frac{\partial z}{\partial y}=\frac{2y}{x^2+y^2}$.

3048. $\frac{\partial z}{\partial x}=-\frac{2}{\sqrt{x^2+y^2}}$; $\frac{\partial z}{\partial y}=\frac{2x}{y\sqrt{x^2+y^2}}$.

3049. $\frac{\partial z}{\partial x}=\frac{xy\sqrt{2}}{(x^2+y^2)\sqrt{x^2-y^2}}$; $\frac{\partial z}{\partial y}=-\frac{x^2\sqrt{2}}{(x^2+y^2)\sqrt{x^2-y^2}}$.

3050. $\frac{\partial z}{\partial x}=\frac{2}{y\sin\frac{2x}{y}}$; $\frac{\partial z}{\partial y}=-\frac{2x}{y^2\sin\frac{2x}{y}}$.

3051. $\frac{\partial z}{\partial x}=-\frac{1}{y}e^{-\frac{x}{y}}$; $\frac{\partial z}{\partial y}=\frac{x}{y^2}e^{-\frac{x}{y}}$.

3052. $\frac{\partial z}{\partial x}=\frac{1}{x+\ln y}$; $\frac{\partial z}{\partial y}=\frac{1}{y(x+\ln y)}$.

3053. $\frac{\partial u}{\partial v}=-\frac{w}{v^2+w^2}$; $\frac{\partial u}{\partial w}=\frac{v}{v^2+w^2}$.

3054. $\frac{\partial z}{\partial x}=\frac{1}{y}\cos\frac{x}{y}\cos\frac{y}{x}+\frac{y}{x^2}\sin\frac{x}{y}\sin\frac{y}{x}$;

$\frac{\partial z}{\partial y}=-\frac{x}{y^2}\cos\frac{x}{y}\cos\frac{y}{x}-\frac{1}{x}\sin\frac{x}{y}\sin\frac{y}{x}$.

3055. $\frac{\partial z}{\partial x}=\frac{y}{x^2}3^{-\frac{y}{x}}\ln 3$; $\frac{\partial z}{\partial y}=-\frac{1}{x}{}^{-\frac{y}{x}}\ln 3$.

3056. $\frac{\partial z}{\partial x}=y^2(1+xy)^{y-1}$; $\frac{\partial z}{\partial y}=xy(1+xy)^{y-1}+(1+xy)^y\ln(1+xy)$.

3057. $\frac{\partial z}{\partial x}=y\ln(x+y)+\frac{xy}{x+y}$; $\frac{\partial z}{\partial y}=x\ln(x+y)+\frac{xy}{x+y}$.

3058. $\frac{\partial z}{\partial x}=x^{x^y}x^{y-1}(y\ln x+1)$; $\frac{\partial z}{\partial y}=x^y x^{x^y}\ln^2 x$

3059. $\frac{\partial u}{\partial x}=yz$; $\frac{\partial u}{\partial y}=xz$; $\frac{\partial u}{\partial z}=xy$.

3060. $\frac{\partial u}{\partial x}=y+z$; $\frac{\partial u}{\partial y}=x+z$; $\frac{\partial u}{\partial z}=x+y$.

3061. $\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}$; $\frac{\partial u}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}}$; $\frac{\partial u}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}$.

3062. $\frac{\partial u}{\partial x}+3x^2+3y-1$; $\frac{\partial u}{\partial y}=z^2+3x$; $\frac{\partial u}{\partial z}=2yz+1$.

3063. $\frac{\partial w}{\partial x}=yz+vz+vy$; $\frac{\partial w}{\partial y}=xz+zv+vx$; $\frac{\partial w}{\partial z}=xy+yv+vx$; $\frac{\partial w}{\partial v}=yz+xz+xy$.

3064. $\frac{\partial u}{\partial x}=(3x^2+y^2+z^2)e^{x(x^2+y^2+z^2)}$;

$\frac{\partial u}{\partial y}=2xye^{x(x^2+y^2+z^2)}$; $\frac{\partial u}{\partial z}=2xz\,e^{x(x^2+y^2+z^2)}$.

3065. $\frac{\partial u}{\partial x}=2x\cos(x^2+y^2+z^2)$; $\frac{\partial u}{\partial y}=2y\cos(x^2+y^2+z^2)$;

$\frac{\partial u}{\partial z}=2z\cos(x^2+y^2+z^2)$.

3066. $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=\frac{\partial u}{\partial z}=\frac{1}{x+y+z}$.

3067. $\frac{\partial u}{\partial x}=\frac{y}{z}x^{\frac{y}{z}-1}$; $\frac{\partial u}{\partial y}=\frac{1}{z}x^{\frac{y}{z}}\ln x$; $\frac{\partial u}{\partial z}=-\frac{y}{z^2}x^{\frac{y}{z}}\ln x$.

3068. $\frac{\partial u}{\partial x}=y^z x^{y^z-1}$; $\frac{\partial u}{\partial y}=zy^{z-1}x^{y^z}\ln x$; $\frac{\partial u}{\partial z}=y^z x^{y^z}\ln x\ln y$.

3069. $\frac{2}{5}$, $\frac{1}{5}$. **3070.** 0, $\frac{1}{4}$.

3071. $\frac{\partial z}{\partial x}=2(2x+y)^{2x+y}[1+\ln(2x+y)]$;

$\frac{\partial z}{\partial y}=(2x+y)^{2x+y}[1+\ln(2x+y)]$.

3072. $\frac{\partial z}{\partial x}=\frac{3}{x\ln y}\left(1+\frac{\ln x}{\ln y}\right)^2$; $\frac{\partial z}{\partial y}=-\frac{3\ln x}{y\ln^2 y}\left(1+\frac{\ln x}{\ln y}\right)^2$.

3073. $\frac{\partial z}{\partial x}=ye^{\sin\pi xy}(1+\pi xy\cos\pi xy)$;

$\frac{\partial z}{\partial y}=xe^{\sin\pi xy}(1+\pi xy\cos\pi xy)$.

3074. $\frac{\partial z}{\partial x}=\frac{1-x^2-y^2-\sqrt{x^2+y^2}}{(1+\sqrt{x^2+y^2})^2}2x$;

$\frac{\partial z}{\partial y}=\frac{1-x^2-y^2-\sqrt{x^2+y^2}}{(1+\sqrt{x^2+y^2})^2}2y$.

3075. $\frac{\partial z}{\partial x}=\frac{y\sqrt{x^y}}{2x(1+x^y)}$; $\frac{\partial z}{\partial y}=\frac{\sqrt{x^y}\ln x}{2(1+x^y)}$.

3076. $\frac{\partial z}{\partial x}=-\frac{y}{(1+\sqrt{xy})\sqrt{xy-x^2y^2}}$; $\frac{\partial z}{\partial y}=-\frac{x}{(1+\sqrt{xy})\sqrt{xy-x^2y^2}}$.

3077. $\frac{\partial z}{\partial x}=\frac{y^2+2xy}{\sqrt{1+(xy^2+yx^2)^2}}$; $\frac{\partial z}{\partial y}=\frac{x^2+2xy}{\sqrt{1+(xy^2+yx^2)^2}}$.

3078. $\frac{\partial z}{\partial x}=-\frac{1}{x^2}\sqrt{\frac{xy-x-y}{xy+x+y}}$; $\frac{\partial z}{\partial y}=-\frac{1}{y^2}\sqrt{\frac{xy-x-y}{xy+x+y}}$.

3079. $$\frac{\partial z}{\partial x}=\frac{y\left[\left(1+\operatorname{arctg}^2\frac{y}{x}\right)^2+2\operatorname{arctg}^3\frac{y}{x}\right]}{(x^2+y^2)\left(1+\operatorname{arctg}^2\frac{y}{x}\right)\left(1+\operatorname{arctg}\frac{y}{x}\right)^2};$$

$$\frac{\partial z}{\partial y}=-\frac{x\left[\left(1+\operatorname{arctg}^2\frac{y}{x}\right)^2+2\operatorname{arctg}^3\frac{y}{x}\right]}{(x^2+y^2)\left(1+\operatorname{arctg}^2\frac{y}{x}\right)\left(1+\operatorname{arctg}\frac{y}{x}\right)^2}.$$

3080. $\frac{\partial u}{\partial x}=-\frac{4kx}{(x^2+y^2+z^2)^3}$;

$\frac{\partial u}{\partial y}=-\frac{4ky}{(x^2+y^2+z^2)^3}$;

$\frac{\partial u}{\partial z}=-\frac{4kz}{(x^2+y^2+z^2)^3}$.

3081. $\frac{\partial u}{\partial x}=\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$; $\frac{\partial u}{\partial y}=-\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$; $\frac{\partial u}{\partial z}=\frac{(x-y)^z\ln(x-y)}{1+(x-y)^{2z}}$.

3082. $\frac{\partial u}{\partial x}=yz(\sin x)^{yz-1}\cos x$; $\frac{\partial u}{\partial y}=z(\sin x)^{yz}\ln\sin x$;

$\frac{\partial u}{\partial z}=y(\sin x)^{yz}\ln\sin x$.

3083. $\frac{\frac{\partial u}{\partial x}}{x}=\frac{\frac{\partial u}{\partial y}}{y}=\frac{\frac{\partial u}{\partial z}}{z}=\frac{2}{r(r^2-1)}$, gde $r=\sqrt{x^2+y^2+z^2}$.

3084. $\frac{\partial w}{\partial x}=(2xy^2-yzv)\operatorname{tg}^3\alpha$; $\frac{\partial w}{\partial y}=(2x^2y-xzv)\operatorname{tg}^3\alpha$; $\frac{\partial w}{\partial z}=(2zv^2-xyv)\operatorname{tg}^3\alpha$;

$\frac{\partial w}{\partial v}=(2z^2v-xyz)\operatorname{tg}^3\alpha$, gde je $\alpha=x^2y^2+z^2v^2-xyzv$.

3085. 4. **3086.** $\left(\frac{\partial u}{\partial z}\right)_{\substack{z=b\\t=a}}=\frac{3b}{2}\sqrt{\frac{ab}{b^2-a^2}}$;

$\left(\frac{\partial u}{\partial t}\right)_{\substack{z=b\\t=a}}=-\frac{3a}{3}\sqrt{\frac{ab}{b^2-a^2}}$;

3087. 1 i —1. **3088.** $\frac{\sqrt{2}}{2}$. **3089.** $\frac{3}{2}$. **3090.** $\frac{13}{22}$. **3091.** 45°.

3092. 30°. **3093.** $\operatorname{arctg}\frac{4}{7}$.

3094. $d_x z=(y^3-6xy^2)\,dx$; $d_y z=(3xy^2-6x^2y+8y^3)\,dy$.

3095. $d_x z=\frac{x\,dx}{\sqrt{x^2+y^2}}$; $d_y z=\frac{y\,dy}{\sqrt{x^2+y^2}}$.

3096. $d_x z=\frac{y(y^2-x^2)\,dx}{(x^2+y^2)^2}$; $d_y z=\frac{x(x^2-y^2)\,dy}{(x^2+y^2)^2}$.

3097. $d_x u=\frac{3x^2\,dx}{x^3+2y^3-z^3}$; $d_y u=\frac{6y^3\,dy}{x^3+2y^3-z^3}$; $d_z u=\frac{-3z^2\,dz}{x^3+2y^3-z^3}$.

3098. $\frac{1}{270}$. **3099.** $\approx 0{,}0187$. **3100.** $\frac{97}{600}$.

3101. $xy\,[(2y^3-3xy^2+4x^2y)\,dx+(4y^2x-3yx^2+2x^3)\,dy]$.

3102. $\frac{x\,dx+y\,dy}{x^2+y^2}$. **3103.** $\frac{2(x\,dy-y\,dx)}{(x-y)^2}$. **3104.** $\frac{y\,dx-x\,dy}{y\sqrt{y^2-x^2}}$

3105. $(x\,dy+y\,dx)\cos(xy)$. **3106.** $\frac{dx}{1+x^2}+\frac{dy}{1+y^2}$.

3107. $\frac{4xy(x\,dy-y\,dx)}{(x^2-y^2)^2}$. **3108.** $\frac{x\,dy+y\,dx}{1+x^2y^2}$.

3109. $x^{xy-1}(yz\,dx+zx\ln x\,dx+xy\ln x\,dz)$,

3124. $e^{\sin t-2t^3}(\cos t-6t^2)$. **3125.** $\sin 2t+2e^{2t}+e^t(\sin t+\cos t)$.

3126. $\frac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}$. **3127.** $\frac{\partial z}{\partial u}=3u^2\sin v\cos v(\cos v-\sin v)$;

$\frac{\partial z}{\partial v}=u^3(\sin v+\cos v)(1-3\sin v\cos v)$.

3128. $\frac{\partial z}{\partial u}=2\frac{u}{v^2}\ln(3u-2v)+\frac{3u^2}{v^2(3u-2v)}$;

$\frac{\partial z}{\partial v}=-\frac{2u^2}{v^3}\ln(3u-2v)-\frac{2u^2}{v^2(3u-2v)}$.

3129. $\frac{\partial u}{\partial x}=\frac{e^x}{e^x+e^y}$; $\frac{du}{dx}=\frac{e^x+3e^{x^3}x^2}{e^x+e^{x^3}}$.

3130. $\frac{dz}{dx}=\frac{e^x(x+1)}{1+x^2e^{2x}}$. **3131.** $\frac{du}{dx}=\frac{1}{1+x^2}$.

3132. $\frac{dz}{dt}=\left(3-\frac{4}{t^3}-\frac{1}{2\sqrt{t}}\right)\sec^2\left(3t+\frac{2}{t^2}-\sqrt{t}\right)$.

3133. $\frac{du}{dx}=e^{ax}\sin x$. **3134.** $dz=\frac{y^2\,dx+x^2\,dy}{(x+y)^2}\operatorname{arctg}(xy+x+y)+\frac{xy\,[(y+1)\,dx+(x+1)\,dy]}{(x+y)\,[1+(xy+x+y)^2]}$.

3135. $\frac{e^{\frac{x^2+y^2}{xy}}}{x^2y^2}[(y^4-x^4+2xy^3)\,x\,dy+(x^4-y^4+2x^3y)\,y\,dx]$.

3136. $$\left.\begin{aligned}&\frac{\partial z}{\partial x}=2x\frac{\partial f}{\partial u}+ye^{xy}\frac{\partial f}{\partial v}.\\&\frac{\partial z}{\partial y}=-2y\frac{\partial f}{\partial u}+xe^{xy}\frac{\partial f}{\partial v}\end{aligned}\right\}\begin{aligned}&u=x^2-y^2;\\&v=e^{xy}.\end{aligned}$$

3185. $\frac{\partial^2 z}{\partial x^2}=\frac{2x^2+y^2}{\sqrt{x^2+y^2}}$; $\frac{\partial^2 z}{\partial y^2}=\frac{x^2+2y^2}{\sqrt{x^2+y^2}}$; $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{xy}{\sqrt{x^2+y^2}}$.

3186. $\frac{\partial^2 z}{\partial x^2}=-\frac{x}{(x^2+y^2)^{\frac{3}{2}}}$; $\frac{\partial^2 z}{\partial y^2}=\frac{x^3+(x^2-y^2)\sqrt{x^2+y^2}}{(x^2+y^2)^{\frac{3}{2}}(x+\sqrt{(x^2+y^2)^2}}$;

$\frac{\partial^2 z}{\partial x\partial y}=-\frac{y}{(x^2+y^2)^{\frac{3}{2}}}$.

3187. $\frac{\partial^2 z}{\partial x^2}=-\frac{2x}{(1+x^2)^2}$; $\frac{\partial^2 z}{\partial y^2}=-\frac{2y}{(1+y^2)^2}$; $\frac{\partial^2 z}{\partial x\,\partial y}=0$.

3188. $\frac{\partial^2 z}{\partial x^2}=2a^2\cos 2(ax+by)$; $\frac{\partial^2 z}{\partial y^2}=2b^2\cos 2(ax+by)$;

$\frac{\partial^2 z}{\partial x\,\partial y}=2ab\cos 2(ax+by)$.

3189. $\frac{\partial^2 z}{\partial x^2}=e^{xe^y+2y}$; $\frac{\partial^2 z}{\partial y^2}=x(1+xe^y)\,e^{xe^y+y}$; $\frac{\partial^2 z}{\partial x\,\partial y}=(1+xe^y)\,e^{xe^y+y}$.

3190. $\frac{\partial^2 z}{\partial x^2} = -\frac{4y}{(x+y)^3}$; $\frac{\partial^2 z}{\partial y^2} = -\frac{4x}{(x+y)^3}$; $\frac{\partial^2 z}{\partial x \partial y} = \frac{2(x-y)}{(x+y)^3}$.

3191. $\frac{\partial^2 z}{\partial x^2} = \frac{\ln y(\ln y+1)}{x^2} e^{\ln x \ln y}$; $\frac{\partial^2 z}{\partial y^2} = \frac{\ln x(\ln x-1)}{y^2} e^{\ln x \ln y}$;

$\frac{\partial^2 z}{\partial x \partial y} = \frac{\ln x \ln y+1}{xy} e^{\ln x \ln y}$.

3192. $\frac{\partial^2 z}{\partial x^2} = \frac{xy^3}{\sqrt{(1-x^2y^2)^3}}$; $\frac{\partial^2 z}{\partial y^2} = \frac{\partial^3 y}{\sqrt{(1-x^2y^2)^3}}$; $\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{\sqrt{(1-x^2y^2)^3}}$.

3193. $\frac{(x-z)y}{\sqrt{(x^2+y^2+z^2-2xz)^3}}$. **3194.** $2y^3(2+xy^2))e^{xy^2}$.

3195. $\frac{4x(3y^2-x^2)}{(x^2+y^2)^3}$. **3196.** $-x(2\sin xy + xy\cos xy)$.

3197. $(x^2y^2z^2+3xyz+1)e^{xyz}$.

3198. $mn(n-1)(n-2)p(p-1)x^{m-1}y^{n-3}z^{p-2}$. **3204.** $a=-3$.

3209.
$$\frac{d^2y}{dx^2} = -\frac{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2 - 2\frac{\partial^2 f}{\partial x \partial y}\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left(\frac{\partial f}{\partial y}\right)^3} = \frac{1}{\left(\frac{\partial f}{\partial y}\right)^3}\begin{vmatrix} 0 & \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial f}{\partial y} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

3219. $-2y\,dx^2+4(y-x)\,dx\,dy+2x\,dy^2$. **3220.** $-\frac{(dx-dy)^2}{(x-y)^2}$.

3221. $\frac{(3x^2-y^2)\,dx^2+8xy\,dx\,dy+(3y^2-x^2)\,dy^2}{(x^2+y^2)^3}$.

3222. $2\sin 2y\,dx\,dy+2x\cos 2y\,dy^2$. **3223.** $e^{xy}[(y\,dx+y\,dy]^2+2d\,x\,dy]$.

3224. $2(z\,dx\,dy+y\,dx\,dz+x\,dy\,dz)$.

3225. $-\cos(2x+y)(2dx+dy)^3$; $(2dx+dy)^3$; 0.

3226. $-\sin(x+y+z)(dx+dy+dz)^2$.

3227. $-\frac{c^4}{z^3}\left[\left(\frac{x^2}{a^2}+\frac{z^2}{c^2}\right)\frac{dx^2}{a^2}+\frac{2xy}{a^2b^2}dx\,dy+\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)\frac{dy^2}{b^2}\right]$.

3228. $\frac{2z[xy^3\,dx^2+(x^2y^2+2xyz^2-z^4)\,dx\,dy+x^3y\,dy^2]}{(z^2-xy)^3}$.

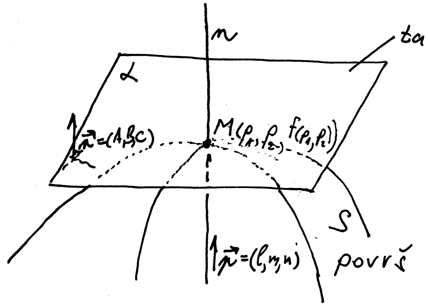
3229. $-31{,}5\,dx^2+206\,dx\,dy-306\,dy^2$. **3230.** $\frac{d^2y}{dt^2}+y$.

Jednačina tangentne ravni i jednačina normale na površ

Jednačina tangentne ravni (hiperravni) na površ S, čija je jednačina $z=f(x_1,x_2)$, u tački $M(\rho_1,\rho_2,f(\rho_1,\rho_2))$ (ako je f diferencijabilna u tački (ρ_1,ρ_2) glasi:

$$z-f(\rho_1,\rho_2)=f'_{x_1}(\rho_1,\rho_2)(x_1-\rho_1)+f'_{x_2}(\rho_1,\rho_2)(x_2-\rho_2)$$

Može li se uspostaviti sličnost sa jednačinom tangente na krivu liniju $y=f(x)$ u ravni?



$Ax+By+Cz+D=0$

$M(\rho_1,\rho_2,f(\rho_1,\rho_2))$ tačka dodira

n - normala na površ $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

Jednačina normale na površ $z=f(x,y)$ u tački $M(\rho_1,\rho_2,f(\rho_1,\rho_2))$ (ako je f diferencijabilna u (ρ_1,ρ_2)) glasi:

$$\frac{x-\rho_1}{f'_x(\rho_1,\rho_2)}=\frac{y-\rho_2}{f'_y(\rho_1,\rho_2)}=\frac{z-f(\rho_1,\rho_2)}{-1}$$

sličnost sa krivom $y=f(x)$ u ravni:
$k_1\cdot k_2=-1$, $M(\rho_1,\rho_2)$ $y-\rho_2=f'(\rho_1)(x-\rho_1)$
$k_2=\frac{-1}{k_1}$, $y-\rho_2=\frac{-1}{f'(\rho_1)}(x-\rho_1)$
$\frac{x-\rho_1}{f'(\rho_1)}=\frac{y-\rho_2}{-1}$

Ako površ S ima jednačinu u implicitnom obliku $F(x,y,z)=0$

$$\alpha:\ F'_x(\rho_1,\rho_2,f(\rho_1,\rho_2))(x-\rho_1)+F'_y(\rho_1,\rho_2,f(\rho_1,\rho_2))(y-\rho_2)+F'_z(\rho_1,\rho_2,f(\rho_1,\rho_2))(z-f(\rho_1,\rho_2))=0$$

$$n:\ \frac{x-\rho_1}{F'_x(\rho_1,\rho_2,f(\rho_1,\rho_2)}=\frac{y-\rho_2}{F'_y(\rho_1,\rho_2,f(\rho_1,\rho_2))}=\frac{z-f(\rho_1,\rho_2)}{F'_z(\rho_1,\rho_2,f(\rho_1,\rho_2))}$$

(#) Naći jednačinu tangentne ravni i normale na površ

a) $z = \frac{x^2}{2} - y^2$ u tački $M(2,-1,1)$

b) $3xyz - z^3 = a^3$ u tački za koju je $x=0$, $y=a$

c)$^{\vee}$ $z = x^2 + 2y^2$ u tački $A(1,1,3)$

d)$^{\vee}$ $z = \operatorname{arc\,tg} \frac{y}{x}$ u tački $(1, 1, \frac{\pi}{4})$ ↗ rj. $\alpha: z = \frac{\pi}{4} - \frac{1}{2}(x-y)$, $n: \frac{x-1}{1} = \frac{y-1}{-1} = \frac{z-\frac{\pi}{4}}{2}$

e)$^{\vee}$ $z = \sqrt{169 - x^2 - y^2}$ ↗ rj. u tački $(3,4,z(3,4))$ $\alpha: 3x + 4y + 12z - 169 = 0$, $n: \frac{x-3}{3} = \frac{y-4}{4} = \frac{z-12}{12}$

f)$^{\vee}$ $\frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{8} = 0$ u tački $M(4,3,4)$ ↗ rj. $\alpha: 3x+4y-6z=0$, $n: \frac{x-4}{3} = \frac{y-3}{4} = \frac{z-4}{-6}$

g)$^{\vee}$ $x^2 + y^2 + z^2 = 2Rz$ u tački $(R\cos\alpha, R\sin\alpha, R)$ $(R>0)$.

Rj. a) $z = f(x,y)$, $z - f(\rho_1,\rho_2) = f'_x(\rho_1,\rho_2)(x-\rho_1) + f'_y(\rho_1,\rho_2)(y-\rho_2)$ jedn. tang. ravni

$z = \frac{x^2}{2} - y^2$, $z'_x = x$, $z'_x(2,-1) = 2$, $\frac{\partial z}{\partial y} = -2y$, $z'_y(2,-1) = 2$

$M(2,-1,1)$, $f(2,-1) = 1$

$z - 1 = 2(x-2) + 2(y+1)$

$2x + 2y - z - 1$ jednačina tangentne ravni

$$\frac{x-\rho_1}{f'_x(\rho_1,\rho_2)} = \frac{y-\rho_2}{f'_y(\rho_1,\rho_2)} = \frac{z-f(\rho_1,\rho_2)}{-1} \Rightarrow \frac{x-2}{2} = \frac{y+1}{2} = \frac{z-1}{-1}$$ jedn. normale

b) Nađimo tačku dodira tangentne ravni i površi

$x=0$, $y=a$, $3xyz - z^3 = a^3 \Rightarrow -z^3 = a^3 \Rightarrow z = -a$

Tačka dodira je $M(0,a,-a)$

$F'_x = 3yz \Rightarrow F'_x(0,a,-a) = -3a^2$

$F'_y = 3xz \Rightarrow F'_y(0,a,-a) = 0$

$F'_z = 3xy - 3z^2 \Rightarrow F'_z(0,a,-a) = -3a^2$

$\alpha: F'_x(\rho_1,\rho_2, f(\rho_1,\rho_2))(x-\rho_1) + F'_y(\rho_1,\rho_2, f(\rho_1,\rho_2))(y-\rho_2) + F'_z(\rho_1,\rho_2, f(\rho_1,\rho_2))(z - f(\rho_1,\rho_2))] = 0$

$-3a^2(x-0) + 0(y-a) + (-3a^2)(z-(-a)) = 0 \Rightarrow -3a^2x - 3a^2z - 3a^3 = 0$

tj. $x + z + a = 0$ jedn. tang. ravni

$$\frac{x-0}{-3a^2} = \frac{y-a}{0} = \frac{z+a}{-3a^2} \Rightarrow \frac{x}{1} = \frac{y-a}{0} = \frac{z+a}{1}$$ jednačina normale

c)$^{\vee}$ rj. $\alpha: 2x + 4y - z - 3 = 0$
$n: \frac{x-1}{2} = \frac{y-1}{4} = \frac{z-3}{-1}$

g)$^{\vee}$ rj. $\alpha: x\cos\alpha + y\sin\alpha - R = 0$
$n: \frac{x - R\cos\alpha}{\cos\alpha} = \frac{y - R\sin\alpha}{\sin\alpha} = \frac{z-R}{0}$

(#) Na površ $x^2 + 2y^2 + 3z^2 = 21$ postaviti tangentnu ravan paralelnu ravni $x + 4y + 6z = 0$.

Rj. $\beta: Ax + By + Cz + D = 0$

$\beta: ?$ $\alpha \parallel \beta$

$\alpha: x + 4y + 6z = 0$

$\vec{n}_\alpha = (1,4,6)$, $\vec{n}_\beta \parallel \vec{n}_\alpha$

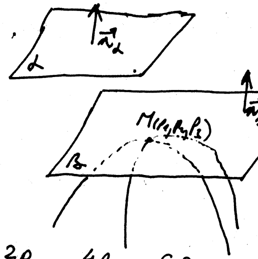
Treba nam tačka dodira tražene tangentne ravni sa površi $x^2 + 2y^2 + 3z^2 = 21$.

$F'_x(\rho_1,\rho_2,\rho_3)(x-\rho_1) + F'_y(\rho_1,\rho_2,\rho_3)(y-\rho_2) + F'_z(\rho_1,\rho_2,\rho_3)(z-\rho_3) = 0$

$F'_x = 2x$

$F'_y = 4y$

$F'_z = 6z$



$$n: \frac{x-\rho_1}{F'_x(\rho_1,\rho_2,\rho_3)} = \frac{y-\rho_2}{F'_y(\rho_1,\rho_2,\rho_3)} = \frac{z z_0}{F'_z(\rho_1,\rho_2,\rho_3)}$$

Vektor normale tražene tangentne ravni je
$\vec{n}_\beta = (2\rho_1, 4\rho_2, 6\rho_3)$

$\vec{n}_\alpha \parallel \vec{n}_\beta \Rightarrow \frac{2\rho_1}{1} = \frac{4\rho_2}{4} = \frac{6\rho_3}{6} \Rightarrow 2\rho_1 = \rho_2 = \rho_3$

odredimo ρ_1, ρ_2 i ρ_3

$\rho_1^2 + 2\cdot 4\rho_1^2 + 3\cdot 4\rho_1^2 = 21$

$21\rho^2 = 21$

$\rho_1 = \pm 1 \Rightarrow \rho_2 = \rho_3 = \pm 2$

I rješenje:

$\rho_1 = -1$, $\rho_2 = \rho_3 = -2$

$-2(x+1) - 8(y+2) - 12(z+2) = 0$

$-2x - 8y - 12z = 42$

$x + 4y + 6z = -21$

II rješenje $\rho_1 = 1$, $\rho_2 = \rho_3 = 2$

$2(x-1) + 8(y-2) + 12(z-2) = 0$

$2x + 8y + 12z - 42 = 0$ $/:2$

$x + 4y + 6z = 21$

jednačin tražene tangentne ravni

ⓣ Odrediti jednačine normale i jednačinu tangentne ravni površi $z=\sqrt{169-x^2-y^2}$ u tački $(3,4,\ z(3,4))$.

Rj:

$$z(3,4)=\sqrt{169-9-16}=\sqrt{144}=12$$

$$M(3,4,12)$$

jednačina tangentne ravni i normale na površ $z=f(x,Y)$ u tački $M(\rho_1,\rho_2,\rho_3)$: $z-\rho_3=z'_x(\rho_1,\rho_2)(x-\rho_1)+z'_Y(\rho_1,\rho_2)(Y-\rho_2)$

$$\frac{x-\rho_1}{z'_x(\rho_1,\rho_2)}=\frac{Y-\rho_2}{z'_Y(\rho_1,\rho_2)}=\frac{z-\rho_3}{-1}$$

$$=\frac{1}{2\sqrt{169-x^2-y^2}}(-2x)=\frac{-x}{\sqrt{169-x^2-y^2}} \Rightarrow z'_x(3,4)=\frac{-3}{\sqrt{169-25}}=\frac{-3}{12}=-\frac{1}{4}$$

$$\frac{\partial z}{\partial Y}=\frac{1}{2\sqrt{169-x^2-y^2}}(-2Y)=\frac{-Y}{\sqrt{169-x^2-y^2}} \Rightarrow z'_Y(3,4)=\frac{-4}{12}=-\frac{1}{3}$$

$$z-12=-\frac{1}{4}(x-3)-\frac{1}{3}(Y-4) \quad /\cdot 12$$

$$12z-144=-3(x-3)-4(Y-4)$$

$$3x+4Y+12z-144-9-16=0$$

$$3x+4Y+12z-169=0$$ jednačina tangentne ravni na površ z

$$\frac{x-3}{-\frac{1}{4}}=\frac{Y-4}{-\frac{1}{3}}=\frac{z-12}{-1} \quad /\cdot\left(\frac{1}{-12}\right)$$

$$\frac{x-3}{3}=\frac{Y-4}{4}=\frac{z-12}{12}$$ jednačina normale na površ z

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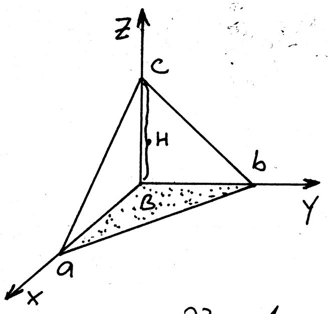
ⓣ Dokazati da tangentne ravni površi $z=\frac{1}{xY}$ tvore s koordinatnim ravnima piramide konstantne zapremine.

Rj. Jednačina tangentne ravni na površ $z=f(x,Y)$ u tački $M(\rho_1,\rho_2,\rho_3)$: $z-\rho_3=z'_x(\rho_1,\rho_2)(x-\rho_1)+z'_Y(\rho_1,\rho_2)(Y-\rho_2)$



$\frac{x}{a}+\frac{Y}{b}+\frac{Z}{c}=1$ kanonični oblik jednačine ravni gdje su a, b i c odsječci koje ravan odsjeca na koordinatnim osama

$$V_{piramide}=\frac{B\cdot H}{3}=\frac{\frac{a.b}{2}\cdot c}{3}=\frac{a.b.c}{6}$$

$$\frac{\partial z}{\partial x}=\frac{1}{Y}\cdot\frac{-1}{x^2}=\frac{-1}{x^2Y} \Rightarrow z'_x(\rho_1,\rho_2)=\frac{-1}{\rho_1^2\rho_2}$$

$$\frac{\partial z}{\partial Y}=\frac{1}{x}\cdot\frac{-1}{Y^2}=\frac{-1}{xY^2} \Rightarrow z'_Y(\rho_1,\rho_2)=\frac{-1}{\rho_1\rho_2^2}$$

$$\rho_3=f(\rho_1,\rho_2)=\frac{1}{\rho_1\cdot\rho_2}$$

$$z-\frac{1}{\rho_1\cdot\rho_2}=\frac{-1}{\rho_1^2\rho_2}(x-\rho_1)+\frac{-1}{\rho_1\rho_2^2}(Y-\rho_2) \quad /\cdot\rho_1^2\rho_2^2$$

$$\rho_1^2\rho_2^2z-\rho_1\rho_2=-\rho_2(x-\rho_1)-\rho_1(Y-\rho_2)$$

$$\rho_1^2\rho_2^2z+\rho_2x+\rho_2Y=\rho_1\rho_2+\rho_1\rho_2+\rho_1\rho_2 \quad /\cdot\frac{1}{\rho_1\rho_2}$$

$$\frac{x}{\rho_1}+\frac{Y}{\rho_1}+\rho_1\rho_2 z=3 \quad /\cdot\frac{1}{3}$$

$$\frac{x}{3\rho_1}+\frac{Y}{3\rho_2}+\frac{z}{\frac{3}{\rho_1\rho_2}}=1 \Rightarrow V_{piramide}=\frac{\overset{1}{\cancel{3}}\rho_1\cdot 3\rho_2\cdot\frac{3}{\rho_1\rho_2}}{\underset{2}{\cancel{6}}}=\frac{9}{2}$$

zapremina piramide za sve tangentne ravni na površ

\# Nađite udaljenost ishodišta koordinatnog sistema od tangentne ravni (helikoida) $Y = x\, tg\frac{z}{a}$ u tački $(a, a, \frac{\pi a}{4})$.

Rj. $F'_x(\rho_1,\rho_2,\rho_3)(x-\rho_1) + F'_Y(\rho_1,\rho_2,\rho_3)(Y-\rho_3) + F'_z(\rho_1,\rho_2,\rho_3)(z-\rho_3) = 0$

jednačina tangentne ravni na površ $F(x,Y,z)=0$.

$$Y - x\, tg\frac{z}{a} = 0$$

$$\frac{\partial F}{\partial x} = -tg\frac{z}{a} \Rightarrow F'_x(a,a,\frac{\pi a}{4}) = -tg\frac{\frac{\pi a}{4}}{a} = -tg\frac{\pi}{4} = -1$$

$$\frac{\partial F}{\partial Y} = 1 \Rightarrow F'_Y(a,a,\frac{\pi a}{4}) = 1$$

$$\frac{\partial F}{\partial z} = \frac{-x}{\cos^2\frac{z}{a}} \cdot \frac{1}{a} = \frac{-x}{a\cos^2\frac{z}{a}} \Rightarrow F'_z(a,a,\frac{\pi a}{4}) = \frac{-a}{a\cos^2\frac{\pi}{4}} = \frac{-1}{\left(\frac{\sqrt{2}}{2}\right)^2}$$

$$F'_z(a,a,\frac{\pi a}{4}) = -2$$

$$-1(x-a) + 1\cdot(Y-a) + (-2)(z - \frac{\pi a}{4}) = 0$$

$$-x + Y - 2z + a - a + \frac{\pi a}{2} = 0$$

$$-x + Y - 2z + \frac{\pi a}{2} = 0$$

jednačina tangentne ravni helikoida u tački $(a, a, \frac{\pi a}{4})$.

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2+B^2+C^2}}, \quad O(0,0,0)$$

$$d = \frac{0+0+0+\frac{\pi a}{2}}{\sqrt{1+1+4}} = \frac{\pi a}{2\sqrt{6}}$$

udaljenost početka koordinatnog sistema od tangentne ravni

\# Napisati jednačinu tangentne ravni i normale na površ $2^{\frac{x}{z}} + 2^{\frac{Y}{z}} = 8$ u tački $M(2,2,1)$.

Rj. Ako površ S ima jednačinu u implicitnom obliku $F(x,Y,z)=0$ tada jednačina tangentne ravni i normale na površ S u tački $M(\rho_1,\rho_2,\rho_3)$ se računaju po formuli:

$$\alpha: F'_x(\rho_1,\rho_2,\rho_3)(x-\rho_1) + F'_Y(\rho_1,\rho_2,\rho_3)(Y-\rho_2) + F'_z(\rho_1,\rho_2,\rho_3)(z-\rho_3) = 0$$

$$n: \frac{x-\rho_1}{F'_x(\rho_1,\rho_2,\rho_3)} = \frac{Y-\rho_2}{F'_Y(\rho_1,\rho_2,\rho_3)} = \frac{z-\rho_3}{F'_z(\rho_1,\rho_2,\rho_3)}$$

$$2^{\frac{x}{z}} + 2^{\frac{Y}{z}} = 8$$

$$\left(\frac{x}{z}\right)'_z = (xz^{-1})'_z = (-1)x\, z^{-2}$$

$$F(x,Y,z) = 2^{\frac{x}{z}} + 2^{\frac{Y}{z}} - 8 = 0$$

$$F'_x = 2^{\frac{x}{z}} \ln 2 \cdot \frac{1}{z} \Rightarrow F'_x(2,2,1) = 4\ln 2.$$

$$F'_Y = 2^{\frac{Y}{z}} \ln 2 \cdot \frac{1}{z} \Rightarrow F'_Y(2,2,1) = 4\ln 2$$

$$F'_z = 2^{\frac{x}{z}} \ln 2 \cdot \left(\frac{x}{z}\right)'_z + 2^{\frac{Y}{z}} \ln 2 \left(\frac{Y}{z}\right)'_z = -\frac{x}{z^2} 2^{\frac{x}{z}} \ln 2 - \frac{Y}{z^2} 2^{\frac{Y}{z}} \ln 2$$

$$= -\frac{1}{z^2} \ln 2 \left(x 2^{\frac{x}{z}} + Y 2^{\frac{Y}{z}}\right)$$

$$F'_z(2,2,1) = -\ln 2(2\cdot 4 + 2\cdot 4) = -16\ln 2$$

$$4\ln 2(x-2) + 4\ln 2(Y-2) + (-16\ln 2)(z-1) = 0.$$

$4x\ln 2 + 4Y\ln 2 - 16z\ln 2 + 8\ln 2 = 0$ jednačina tangentne ravni

$$\frac{x-2}{4\ln 2} = \frac{Y-2}{4\ln 2} = \frac{z-1}{-16\ln 2} \Rightarrow \frac{x-2}{1} = \frac{Y-2}{1} = \frac{z-1}{-4}$$

jednačina normale na površ

(#) Naći jednačinu tangentne ravni elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ koja na koordinatnim osama odsjeca jednake pozitivne odsječke.

Rj: Jednačina tangentne ravni na površ $F(x,y,z)=0$ u tački $M(p_1,p_2,p_3)$ ima jednačinu $F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)$

Nađimo jednačinu tangentne ravni na elipsoid u proizvoljnoj tački $M(p_1,p_2,p_3)$: (U našem slučaju $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$)

$$F'_x=\frac{1}{a^2}\cdot 2x=\frac{2x}{a^2},\quad F'_y=\frac{2y}{b^2},\quad F'_z=\frac{2z}{c^2}$$

$$F'_x(M)=\frac{2p_1}{a^2},\quad F'_y(M)=\frac{2p_2}{b^2},\quad F'_z(M)=\frac{2p_3}{c^2}$$

$$\frac{2p_1}{a^2}(x-p_1)+\frac{2p_2}{b^2}(y-p_2)+\frac{2p_3}{c^2}(z-p_3)=0 \quad /\cdot\frac{1}{2}$$

$$\frac{p_1}{a^2}x+\frac{p_2}{b^2}y+\frac{p_3}{c^2}z=\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}$$

Napišimo jednačinu ravni u kanonskom obliku

$$\frac{x}{\frac{a^2}{p_1}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}+\frac{y}{\frac{b^2}{p_2}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}+\frac{z}{\frac{c^2}{p_3}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}=1$$

Odavde možemo primjetiti da ako želimo da jednačina tangentne ravni na kordanatnim osama odsjeca jednake odsječke, potrebno i dovoljno je da $\frac{a^2}{p_1}=\frac{b^2}{p_2}$, $\frac{a^2}{p_1}=\frac{c^2}{p_3}$ i $\frac{b^2}{p_2}=\frac{c^2}{p_3}$. ...(*)

Isto tako primjetimo da je $\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}=1$ ...(**) (ZAŠTO?)

$$(*)\Rightarrow p_1=\frac{a^2}{b^2}p_2,\quad p_3=\frac{c^2}{b^2}p_2 \quad ...(1)$$

Sad imamo:

$$\frac{x}{\frac{a^2}{\frac{a^2}{b^2}p_2}}+\frac{y}{\frac{b^2}{p_2}}+\frac{z}{\frac{c^2}{\frac{c^2}{b^2}p_2}}=1 \quad /:p_2$$

$$\frac{x}{b^2}+\frac{y}{b^2}+\frac{z}{b^2}=\frac{1}{p_2}$$

Kada (1) stavimo u (**) dobićemo da je

$$p_2=\frac{b^2}{\sqrt{a^2+b^2+c^2}}$$

Prema tome:

$x+y+z=\sqrt{a^2+b^2+c^2}$ je jednačina tražene tangente

Zadaci za vježbu

Površi

U zadacima 3410 — 3419 sastaviti jednčine tangencijalnih ravni i normala za date površi u navedenim tačkama.

3410. $z=2x^2-4y^2$ **u tački (2, 1, 4).**

3411. $z=xy$ **u tački (1, 1, 1)**

3412. $z=\frac{x^3-3axy+y^3}{a^2}$ **u tački** $(a,\ a,\ -a$

3413. $z=\sqrt{x^2+y^2}-xy$ **u tački (3, 4, —7).**

3414. $z=\operatorname{arctg}\frac{y}{x}$ **u tački** $\left(1,\ 1,\ \frac{\pi}{4}\right)$.

3415. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ **u tački** $\left(\frac{a\sqrt{3}}{3},\ \frac{b\sqrt{3}}{3},\ \frac{c\sqrt{3}}{3}\right)$.

3416. $x^3+y^3+z^3+xyz-6=0$ **u tački (1, 2, —1).**

3417. $3x^4-4y^3z+4z^2xy-4z^3x+1=0$ **u tački (1, 1, 1).**

3418. $(z^2-x^2)\,xyz-y^5=5$ **u tački (1, 1, 2).**

3419. $4+\sqrt{x^2+y^2+z^2}=x+y+z$ **u tački (2, 3, 6).**

3420. **pokazati da jednačina tangencijalne ravni elipsoida** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ **u proizvoljnoj tački** $M_0(x_0,\ y_0,\ z_0)$ **glasi:**

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{z_0z}{c^2}=1.$$

3421. Naći tangencijalnu ravan elipsoida $x^2+2y^2+z^2=1$ paralelnu ravni $x-y+2z=0$.

3422. Naći tangencijalnu ravan elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ koja od koordinatnih osa odseca jednake pozitivne odsečke.

3423. Pokazati da se površi $x+2y-\ln z+4=0$ i $x^2-xy-8x+z+5=0$ dodiruju (tj. imaju zajedničku tangencijalnu ravan) u tački (2. —3, 1).

3424. Dokazati da se sve tangencijalne ravni površi $z = xf\left(\frac{y}{x}\right)$ seku u jednoj tački.

3425. Sastaviti jednačine tangencijalne ravni i normalne sfere $r\{u\cos v; u\sin v, \sqrt{a^2-u^2}\}$ u tački $r_0\{x_0, y_0, z_0\}$.

3426. Sastaviti jednačine tangencijalne ravni i normale hiperboličnog paraboloida $r\{a(u+v), b(u-v), uv\}$ u proizvoljnoj tački $r_0\{x_0, y_0, z_0\}$.

3427. Dokazati da su sfere $x^2+y^2+z^2=ax$ i $x^2+y^2+z^2=by$ uzajamno normalne.

3428. Pokazati da tangencijalne ravni površi $xyz=a^3$ u svakoj njenoj tački obrazuju sa koordinatnim ravnima tetraedre konstantne zapremine i naći tu zapreminu.

3429. Pokazati da tangencijalne ravni površi $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$ odsecaju od koordinatnih osa odsečke čiji zbir ima vrednost a.

3430. Za površ $z=xy$ sastaviti jednačinu tangencijalne ravni, normalne na pravu

$$\frac{x+2}{2}=\frac{y+2}{1}=\frac{z-1}{-1}.$$

3431. Pokazati da je za površ $x^2+y^2+z^2=y$ dužina odsečka normale između površi i ravni xOy jednaka rastojanju od koordinatnog početka do prodora normale kroz tu ravan.

3432. Dokazati da normala obrtnog elipsoida $\frac{x^2+z^2}{9}+\frac{y^2}{25}=1$ u svakoj njegovoj tački $P(x, y, z)$ zaklapa jednake uglove sa pravama PA i PB ako je $A(0, -4, 0)$ i $B(0, 4, 0)$.

3433. Dokazati da sve normale obrtne površi $z=f(\sqrt{x^2+y^2})$ presecaju osu obrtanja.

3434. Za površ $x^2-y^2-3z=0$ naći tangencijalnu ravan koja prolazi kroz tačku $A(0, 0, -1)$ i paralelna je pravoj $\frac{x}{2}=\frac{y}{1}=\frac{z}{2}$.

3435. Na sferi $x^2+y^2+z^2-6y+4z=12$ naći tačke u kojima su tangencijalne ravni paralelne koordinatnim ravnima.

3436. Naći tangencijalnu ravan površi $x=u+v$, $y=u^2+v^2$, $z=u^3+v^3$ u proizvoljnoj tački:

a) uzimajući jednačine površi u parametarskom vidu;
b) napisavši jednačinu ove površi u obliku $z=f(x, y)$.

3437. Naći geometrijsko mesto podnožja normala povučenih iz koordinatnog početka na tangencijalne ravni obrtnog paraboloida $2pz=x^2+y^2$.

3438. Naći geometrijsko mesto podnožja normala spuštenih iz koordinatnog početka na tangencijalne ravni površi $xyz=a^3$.

Rješenja

3410. $8x-8y-z=4;\quad \frac{x-2}{8}=\frac{y-1}{-8}=\frac{z-4}{-1}.$

3411. $x+y-z-1=0;\quad \frac{x-1}{1}=\frac{y-1}{1}=\frac{z-1}{-1}.$

3412. $z+a=0,\quad x=a,\quad y=a.$

3413. $17x+11y+5z=60;\quad \frac{x-3}{17}=\frac{y-4}{11}=\frac{z+7}{5}.$

3414. $x-y+2z-\frac{\pi}{2}=0;\quad \frac{x-1}{1}=\frac{y-1}{-1}=\frac{z-\frac{\pi}{2}}{2}.$

3415. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\sqrt{3};$

$$a\left(x-\frac{a\sqrt{3}}{3}\right)=b\left(y-\frac{b\sqrt{3}}{3}\right)=c\left(z-\frac{c\sqrt{3}}{3}\right)$$

3416. $x+11y+5z-18=0;\quad \frac{x-1}{1}=\frac{y-2}{11}=\frac{z+1}{5}.$

3417. $3x-2y-2z+1=0;\quad \frac{x-1}{3}=\frac{y-1}{-2}=\frac{z-1}{-2}.$

3418. $2x+y+11z-25=0;\quad \frac{x-1}{2}=\frac{y-1}{1}=\frac{z-2}{11}.$

3419. $5x+4y+z-28=0;\quad \frac{x-2}{5}=\frac{y-3}{4}=\frac{z-6}{1}.$

3421. $x-y+2z=\sqrt{\frac{11}{2}}$ i $x-y+2z=-\sqrt{\frac{11}{2}}.$

3422. $x+y+z=\sqrt{a^2+b^2+c^2}.$

3424. Sve ravni prolaze kroz koordinatni početak.

3425. $x_0x+y_0y+z_0z=a^2;\quad \frac{x}{x_0}=\frac{y}{y_0}=\frac{z}{z_0}.$

3426. $\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=2(z+z_0);\quad \frac{a(x-x_0)}{bx_0}=-\frac{b(y-y_0)}{ay_0}=\frac{z-z_0}{-2ab}.$

3428. $\frac{9}{2}a^3.$ **3430.** $2x+y-z=2.$ **3434.** $4x-2y-3z=3.$

3435. Paralelna ravni xOy u tačkama $(0, 3, 3)$ i $(0\ 3, -7)$; ravni yOz u tačkama $(5, 3, -2)$ i $(-5, 3, -2)$; ravni xOz u tačkama $(0, -2, -2)$ i $(0, 8, -2)$.

3436. a) $6u_0v_0x-3(u_0+v_0)y+2z+(u_0+v_0)(u_0^2-4u_0v_0+v_0^2)=0;$
b) $3(x_0^2-y_0)x-3x_0(y+y_0)+2z+4z_0=0.$

3437. $2z(x^2+y^2+z^2)+p(x^2+y^2)=0.$ **3438.** $(x^2+y^2+z^2)^3=27a^3xyz.$

Ekstremne vrijednosti f-ja dviju promjenjivih

Neka je data f-ja $z = f(x, y)$.

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= 0 \\ \frac{\partial z}{\partial y} &= 0 \end{aligned}\right\}$$

SISTEM

rješenjem sistema dobijemo stacionarne tačke koje mogu ali i ne moraju biti ekstrem

npr. $M(p_1, p_2)$ je jedna stacionarna tačka.

$$A = \frac{\partial^2 z(p_1,p_2)}{\partial x^2}$$

$$B = \frac{\partial^2 z(p_1,p_2)}{\partial x \partial y}$$

$$C = \frac{\partial^2 z(p_1,p_2)}{\partial y^2}$$

$D = AC - B^2$

$D > 0$ f-ja ima ekstrem u tački $M(p_1, p_2)$

a) $A > 0$ imamo z_{min}

b) $A < 0$ imamo z_{max}

$D < 0$ f-ja nema ekstrem

$D = 0$ potrebno ispitati ponašanje f-je u okolini stacionarne tačke:

$\Delta z(M) = z(p_1+\varepsilon, p_2+\omega) - z(p_1, p_2)$ – priraštaj f-je

$\Delta z \geq 0 \quad \forall\varepsilon; \forall\omega \Rightarrow$ u tački M f-ja ima minimum

$\Delta z \leq 0 \quad \forall\varepsilon; \forall\omega \Rightarrow$ u tački M f-ja ima maksimum

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\# Naći ekstreme f-je $z = x^2 - 2x - y - \ln(2-y) + 4$.

Rj. $\mathcal{D}: 2 - y > 0$

$$\frac{\partial z}{\partial x} = 2x - 2$$

$$\frac{\partial z}{\partial y} = -1 - \frac{1}{2-y}\cdot(-1) = \frac{1}{2-y} - 1$$

$$\begin{aligned} 2x - 2 &= 0 \\ \frac{1}{2-y} - 1 &= 0 \end{aligned}$$

$x = 1, \; y = 1$

Tačka $M(1,1)$ je stacionarna tačka (kandidat za ekstrem)

$(2-y)^{-1}$

$$\frac{\partial^2 z}{\partial x^2} = 2$$

$$\frac{\partial^2 z}{\partial x \partial y} = 0$$

$$\frac{\partial^2 z}{\partial y^2} = (-1)(2-y)^{-2}\cdot(-1) = \frac{1}{(2-y)^2}$$

$M(1,1)$

$A = 2, \; B = 0, \; C = 1$

$D = AC - B^2 = 2 > 0$

F-ja ima ekstrem.

$A > 0 \Rightarrow$ f-ja ima minimum

$z_{min}(1,1) = 1 - 2 - 1 - \ln 1 + 4 = -2 + 4 = 2$

(#) Naći ekstreme f-je $z=x^3-5xy+5y^2+7x-15y$.

Rj: Pronađimo stacionarne tačke

$$\frac{\partial z}{\partial x}=3x^2-5y+7$$

$$\frac{\partial z}{\partial y}=10y-5x-15$$

$$3x^2-5y+7=0 \quad /\cdot 2$$
$$-5x+10y-15=0$$

$$6x^2-10y+14=0$$
$$-5x+10y-15=0 \quad +$$

$$6x^2-5x-1=0$$

$$D=25+24=49$$

$$x_{1,2}=\frac{5\pm7}{2\cdot6}$$

$$x_1=\frac{-2}{2\cdot6}=\frac{-1}{6},\quad x_2=\frac{12}{12}=1$$

$$6\left(x+\frac{1}{6}\right)(x-1)=0$$

Za $x_1=-\frac{1}{6} \Rightarrow -5\cdot\left(-\frac{1}{6}\right)+10y-15=0$

$$10y=15-\frac{5}{6}$$

$$10y=\frac{90-5}{6}=\frac{85}{6}$$

$$y=\frac{\overset{17}{\cancel{85}}}{\underset{2}{\cancel{10}}\,6}=\frac{17}{12}$$

$x_2=1 \Rightarrow -5+10y-15=0$

$$10y=20$$

$$y=2$$

Stacionarne tačke su $(1,2)$ i $\left(-\frac{1}{6},\frac{17}{12}\right)$

$$\frac{\partial^2 z}{\partial x^2}=6x$$

$$\frac{\partial^2 z}{\partial x\partial y}=-5$$

$$\frac{\partial^2 z}{\partial y^2}=10$$

Za $M_1(1,2)$

$A=6,\ B=-5,\ C=10,\ D=AC-B^2=60-25>0$

f-ja ima ekstrem

$A>0 \Rightarrow$ f-ja ima minimum

$z_{min}(1,2)=1-10+20+7-30=8+10-30=8-20=-12$

Za $M_2\left(-\frac{1}{6},\frac{17}{12}\right)$

$A=-1,\ B=-5,\ C=10,\ D=AC-B^2=-10-25=-35$

F-ja u ovoj tački nema ekstrem

(#) Nađi ekstreme f-je $z=x+y-\frac{3}{2}\ln(x^2+y^2+1)$.

Rj: Izračunajmo prve parcijalne izvode

$$\frac{\partial z}{\partial x}=1-\frac{3}{\cancel{2}}\cdot\frac{1}{x^2+y^2+1}\cdot\cancel{2}x=1-\frac{3x}{x^2+y^2+1}$$

$$\frac{\partial z}{\partial y}=1-\frac{3}{\cancel{2}}\cdot\frac{1}{x^2+y^2+1}\cdot\cancel{2}y=1-\frac{3y}{x^2+y^2+1}$$

Nađimo stacionarne tačke

$$1-\frac{3x}{x^2+y^2+1}=0$$
$$1-\frac{3y}{x^2+y^2+1}=0$$

$$3x=3y \Rightarrow x=y$$

$$1-\frac{3x}{2x^2+1}=0 \quad /\cdot 2x^2+1$$

$$2x^2+1-3x=0$$

$$2x^2-3x+1=0$$

$$D=9-8=1$$

$$x_{1,2}=\frac{3\pm1}{4}$$

$$x_1=\frac{2}{4}=\frac{1}{2}$$

$$x_2=1$$

Stacionarne tačke su $M_1\left(\frac{1}{2},\frac{1}{2}\right)$ i $M_2(1,1)$

Nađimo druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2}=0-3\cdot\frac{1\cdot(x^2+y^2+1)-x\cdot2x}{(x^2+y^2+1)^2}=-3\cdot\frac{-x^2+y^2+1}{(x^2+y^2+1)^2}=3\,\frac{x^2-y^2-1}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=0-3x\cdot(-1)(x^2+y^2+1)^{-2}\cdot2y=6\,\frac{xy}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial y^2}=0-3\,\frac{1\cdot(x^2+y^2+1)-y\cdot2y}{(x^2+y^2+1)^2}=-3\,\frac{x^2-y^2+1}{(x^2+y^2+1)^2}$$

Za $M_1\left(\frac{1}{2},\frac{1}{2}\right)$, $A=3\cdot\frac{-1}{\left(\frac{1}{2}+1\right)^2}=\frac{-3}{\frac{9}{4}}=\frac{-12}{9}=-\frac{4}{3}$, $B=\frac{2}{3}$, $C=-\frac{4}{3}$

$D=AC-B^2=\frac{16}{9}-\frac{4}{9}>0$ f-ja ima ekstrem u tački M_1

$A<0$ f-ja ima minimum $z_{min}\left(\frac{1}{2},\frac{1}{2}\right)=1-\frac{3}{2}\ln\frac{3}{2}$

Za $M_2(1,1)$, $A=-\frac{1}{3}$, $B=\frac{2}{3}$, $C=-\frac{1}{3}$

$D=AC-B^2=\frac{1}{9}-\frac{4}{9}<0$ f-ja u tački M_2 nema ekstrem

(#) Naći ekstreme f-je $z = x + y - \frac{3}{2}\ln(x^2+y^2+1)$.

Rj. Pronađimo prve parcijalne izvode

$$\frac{\partial z}{\partial x} = 1 - \frac{3}{2}\cdot\frac{2x}{x^2+y^2+1} = 1 - \frac{3x}{x^2+y^2+1}$$

$$\frac{\partial z}{\partial y} = 1 - \frac{3}{2}\cdot\frac{2y}{x^2+y^2+1} = 1 - \frac{3y}{x^2+y^2+1}$$

Pronađimo stacionarne tačke

$$\left.\begin{array}{l} \frac{\partial z}{\partial x} = 0 \Rightarrow 1 = \frac{3x}{x^2+y^2+1} \\ \frac{\partial z}{\partial y} = 0 \Rightarrow 1 = \frac{3y}{x^2+y^2+1} \end{array}\right\} \Rightarrow x = y \text{ (deljenjem jednačina)}$$

Sad imamo $x=y$ i $1 = \frac{3x}{x^2+y^2+1} \Rightarrow 1 = \frac{3x}{2x^2+1} \Rightarrow 2x^2-3x+1=0$

$D = 9-8 = 1$

$x_1 = 1,\ x_2 = \frac{1}{2}$

Stacionarne tačke su $M_1(1,1)$ i $M_2(\frac{1}{2}, \frac{1}{2})$.

Pronađimo druge parcijalne izvode.

$$\frac{\partial^2 z}{\partial x^2} = \left(1 - \frac{3x}{x^2+y^2+1}\right)'_x = \frac{-3(x^2+y^2+1)+3x\cdot 2x}{(x^2+y^2+1)^2} = \frac{3x^2-3y^2-3}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(1 - \frac{3y}{x^2+y^2+1}\right)'_y = \left|\begin{array}{l}\text{zbog}\\ \text{simetričnosti}\end{array}\right| = \frac{3y^2-3x^2-3}{(x^2+y^2+1)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{3x\cdot 2y}{(x^2+y^2+1)^2} = \frac{6xy}{(x^2+y^2+1)^2}$$

Za tačku $M_1(1,1)$: $A = -\frac{3}{9} = -\frac{1}{3}$, $B = \frac{6}{9} = \frac{2}{3}$, $C = -\frac{3}{9} = -\frac{1}{3}$, $D = AC - B^2$

$D = \frac{1}{9} - \frac{4}{9} < 0 \Rightarrow$ u M_1 f-ja nema ekstrema

Za tačku $M_2(\frac{1}{2}, \frac{1}{2})$: $A = \frac{-3}{(\frac{3}{2})^2} = -\frac{3}{\frac{9}{4}} = -\frac{12}{9} = -\frac{4}{3} \Rightarrow C = -\frac{4}{3}$

$B = \frac{\frac{3}{2}}{\frac{9}{4}} = \frac{12}{18} = \frac{2}{3}$, $D = AC - B^2 = \frac{16}{9} - \frac{4}{9} = \frac{12}{9} = \frac{4}{3} > 0 \Rightarrow$ f-ja u tački M_2 ima ekstrem

$A < 0 \Rightarrow$ u M_2 f-ja ima maximum. $z_{max}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2} + \frac{1}{2} - 3\ln(\frac{1}{4}+\frac{1}{4}+1) = 1 - \ln\frac{27}{8}$

(#) Naći ekstreme f-je $z = \frac{8}{x} + \frac{x^2}{y} + y + 1$.

Rj. Pronađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = 8\cdot(-1)\,x^{-2} + 2\frac{x}{y} = \frac{-8}{x^2} + 2\frac{x}{y}$$

$$\frac{\partial z}{\partial y} = x^2\cdot(-1)\,y^{-2} + 1 = \frac{-x^2}{y^2} + 1$$

$$-\frac{8}{x^2} + \frac{2x}{y} = 0$$

$$-\frac{x^2}{y^2} + 1 = 0$$

$$\frac{8}{x^2} - 2\frac{x}{y} = 0$$

$$\frac{x^2}{y^2} = 1 \Rightarrow \left(\frac{x}{y}\right)^2 = 1$$

Prema tome $\frac{x}{y} = 1$ i $\frac{x}{y} = -1$

Za $\frac{x}{y} = 1 \Rightarrow \frac{8}{x^2} - 2\cdot 1 = 0$

$\frac{8}{x^2} = 2 \quad / \cdot x^2\ (x \neq 0)$

$2x^2 = 8$

$x^2 = 4$

$x_1 = -2,\ x_2 = 2$

$x_1 = -2 \Rightarrow \frac{x}{y} = 1$

$y = -2$

$(-2, -2)$

za $x_2 = 2 \Rightarrow$

$\Rightarrow \frac{x}{y} = 1$

$y_2 = 2$

$(2, 2)$

Za $\frac{x}{y} = -1$ imamo

$\frac{8}{x^2} + 2 = 0$

$\frac{8}{x^2} = -2 \quad / \cdot x^2\ (x \neq 0)$

$-2x^2 = 8$

ova jednačina nema rešenja u skupu realnih brojeva

Stacionarne tačke su $M_1(-2,-2)$ i $M_2(2,2)$.

Nađimo druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2} = (-8)(-2)\,x^{-3} + \frac{2}{y} = \frac{16}{x^3} + \frac{2}{y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x\cdot(-1)\,y^{-2} = \frac{-2x}{y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = -x^2\cdot(-2)\,y^{-3} = \frac{2x^2}{y^3}$$

Za $M_2(2,2)$

$A = 2+1 = 3$, $B = \frac{-4}{4} = -1$, $C = \frac{8}{8} = 1$

$D = AC - B^2 = 3 - 1 = 2 > 0$ f-ja ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

$z_{min}(2,2) = 4 + 2 + 2 + 1 = 9$

Za $M_1(-2,-2)$

$A = \frac{16}{-8} + \frac{2}{-2} = -2 - 1 = -3$

$B = \frac{-2\cdot(-2)}{4} = 1$, $C = \frac{2\cdot 4}{-8} = -1$

$D = AC - B^2 = 3 - 1 = 2 > 0$.

f-ja u tački $M_1(-2,-2)$ ima ekstrem,

$A < 0$ f-ja ima maksimum

$z_{max}(-2,-2) = -4 - 2 - 2 + 1 = -7$

(#) Nađi ekstreme f-je $z=(x^2+y)\sqrt{e^y}$.

Rj. $\frac{\partial z}{\partial x} = 2x\sqrt{e^y}$

$$\frac{\partial z}{\partial y} = \sqrt{e^y} + (x^2+y)\frac{1}{2\sqrt{e^y}}\cdot e^y = \sqrt{e^y} + (x^2+y)\cdot\frac{1}{2}\sqrt{e^y}$$

$$= \left(\frac{1}{2}x^2 + \frac{1}{2}y + 1\right)\sqrt{e^y} = \frac{1}{2}(x^2+y+2)\sqrt{e^y}$$

$$\frac{\begin{aligned} 2x\sqrt{e^y} &= 0 \\ \tfrac{1}{2}(x^2+y^2+1)\sqrt{e^y} &= 0 \end{aligned}}{e^y > 0 \quad \forall y \in \mathbb{R}}$$

prema tome $x=0$

$\sqrt{e^y} > 0 \quad \forall x \in \mathbb{R}$

$x^2+y+2=0$

$x=0 \Rightarrow y+2=0$

$y=-2$

$M(0,-2)$ je stacionarna tačka (kandidat za ekstrem)

$$\frac{\partial^2 z}{\partial x^2} = 2\sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = 2x\frac{1}{2\sqrt{e^y}}\cdot e^y = x\sqrt{e^y}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{2}\sqrt{e^y} + \left(\frac{1}{2}x^2+\frac{1}{2}y+1\right)\frac{e^y\cdot\sqrt{e^y}}{2\sqrt{e^y}\cdot\sqrt{e^y}} = \frac{1}{2}\sqrt{e^y}\left(\frac{1}{2}x^2+\frac{1}{2}y+2\right)$$

$M(0,-2)$

$A = 2\sqrt{e^{-2}} = 2\frac{1}{\sqrt{e^2}}$

$B=0$

$C = \frac{1}{2}\sqrt{e^{-2}}\left(\frac{1}{2}0 + \underbrace{\frac{1}{2}(-2)}_{-1}+2\right) = \frac{1}{2}\sqrt{\frac{1}{e^2}}$

$D = AC-B^2 = \frac{2}{\sqrt{e^2}}\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{e^2}} = \frac{1}{e^2}$

$D>0 \Rightarrow$ f-ja ima ekstrem

$A>0 \Rightarrow$ f-ja ima minimum

$$z_{min}(0,-2) = (0-2)\sqrt{e^{-2}} = (-2)\cdot\frac{1}{\sqrt{e^2}} \approx -0.7358$$

(#) Nađi ekstreme f-je $z = e^{-2x^2}(x-y^2)$.

Rj. Nađimo stacionarne tačke

$$\frac{\partial z}{\partial x} = e^{-2x^2}\cdot(-4)x(x-y^2) + e^{-2x^2}\cdot 1 = e^{-2x^2}(-4x^2+4xy^2+1)$$

$$\frac{\partial z}{\partial y} = e^{-2x^2}\cdot(-2)y = -2ye^{-2x^2}$$

$$\frac{\begin{aligned} e^{-2x^2}(-4x^2+4xy^2+1) &= 0 \\ -2y\,e^{-2x^2} &= 0 \end{aligned}}{\dfrac{\begin{aligned} -4x^2+4xy^2+1 &= 0 \\ -2y &= 0 \end{aligned}}{-4x^2+1=0}} \quad \Rightarrow y=0$$

e^{-2x^2} je uvijek pozitivno

$x^2 = \frac{1}{4} \Rightarrow x_1 = -\frac{1}{2}, \quad x_2 = \frac{1}{2}$

Stacionarne tačke su $M_1(-\frac{1}{2},0)$ i $M_2(\frac{1}{2},0)$

$$\frac{\partial^2 z}{\partial x^2} = e^{-2x^2}\cdot(-4x)(-4x^2+4xy^2+1) + e^{-2x^2}(-8x+4y^2) =$$

$$= e^{-2x^2}(16x^3-16x^2y^2-4x-8x+4y^2) = e^{-2x^2}(16x^3-16x^2y^2-12x+4y^2)$$

$$= 4e^{-2x^2}(4x^3-4x^2y^2-3x+y^2)$$

$$\frac{\partial^2 z}{\partial x \partial y} = e^{-2x^2}(8xy) = 8xy\,e^{-2x^2}$$

$$\frac{\partial^2 z}{\partial y^2} = -2e^{-2x^2}$$

Za tačku $M_1(-\frac{1}{2},0)$

$A = 4e^{-2\cdot\frac{1}{4}}\left(4\cdot\left(-\frac{1}{8}\right) - 4\cdot\frac{1}{4}\cdot 0 - 3\left(-\frac{1}{2}\right) + 0\right) = 4e^{-\frac{1}{2}}\left(-\frac{1}{2}+\frac{3}{2}\right) = \frac{4}{\sqrt{e}}$

$B=0, \quad C = -2e^{-2\cdot\frac{1}{4}} = -2e^{-\frac{1}{2}} = \frac{-2}{\sqrt{e}}$

$D = AC-B^2 = \frac{-8}{e} < 0$

F-ja z u tački M_1 nema ekstrem

Za tačku $M_2(\frac{1}{2},0)$

$A = 4e^{-2\cdot\frac{1}{4}}\left(4\cdot\frac{1}{8} - 0 - 3\cdot\frac{1}{2} + 0\right) =$

$= 4e^{-\frac{1}{2}}\left(\frac{1}{2}-\frac{3}{2}\right) = \frac{-4}{\sqrt{e}}$

$B=0, \quad C = -2e^{-2\cdot\frac{1}{4}} = \frac{-2}{\sqrt{e}}$

$D = AC-B^2 = \frac{8}{e} > 0 \Rightarrow$ f-ja za u tački M_2 ima ekstrem

$A<0 \Rightarrow z_{max}(\frac{1}{2},0) = e^{-2\cdot\frac{1}{4}}\left(\frac{1}{2}-0\right) = \frac{1}{2}\cdot e^{-\frac{1}{2}} = \frac{1}{2\sqrt{e}}$

Odrediti ekstremne vrijednosti f-je

$$Z = \frac{xY}{2} + (47 - x - Y)\left(\frac{x}{3} + \frac{Y}{4}\right).$$

Rj.

$$\frac{\partial z}{\partial x} = \frac{1}{2}Y + (-1)\left(\frac{x}{3} + \frac{Y}{4}\right) + (47 - x - Y)\cdot\frac{1}{3} = \underline{\underline{\frac{1}{2}Y}} - \underline{\frac{1}{3}x} - \underline{\underline{\frac{1}{4}Y}} + \frac{47}{3} \underline{- \frac{1}{3}x} - \underline{\underline{\frac{1}{3}Y}}$$

$$= -\frac{2}{3}x + \frac{6-3-4}{12}Y + \frac{47}{3} = -\frac{2}{3}x - \frac{1}{12}Y + \frac{47}{3}$$

$$\frac{\partial z}{\partial Y} = \frac{1}{2}x + (-1)\left(\frac{x}{3} + \frac{Y}{4}\right) + (47 - x - Y)\cdot\frac{1}{4} = \underline{\frac{1}{2}x} - \underline{\frac{1}{3}x} - \underline{\underline{\frac{1}{4}Y}} + \frac{47}{4} - \underline{\frac{1}{4}x} - \underline{\underline{\frac{1}{4}Y}}$$

$$= -\frac{1}{12}x - \frac{1}{2}Y + \frac{47}{4}$$

$$\begin{array}{ll} -\frac{2}{3}x - \frac{1}{12}Y + \frac{47}{3} = 0 & /\cdot 12 \\ -\frac{1}{12}x - \frac{1}{2}Y + \frac{47}{4} = 0 & /\cdot 12 \\ \hline -8x - Y + 188 = 0 & \\ -x - 6Y + 141 = 0 & \\ \hline -8x - Y + 188 = 0 & \\ x = -6Y + 141 & \\ \hline \end{array}$$

$$-8(-6Y + 141) - Y + 188 = 0$$

$$48Y - 1128 - Y + 188 = 0$$

$$47Y = 940$$

$$Y = 20$$

$$x = -6Y + 141 = -120 + 141 = 21$$

Stacionarna tačka je $M(21,20)$.

$$\frac{\partial^2 z}{\partial x^2} = -\frac{2}{3}$$

$$\frac{\partial^2 z}{\partial x \partial Y} = -\frac{1}{12}$$

$$\frac{\partial^2 z}{\partial Y^2} = -\frac{1}{2}$$

$D = AC - B^2$

$M(21,20)$

$A = -\frac{2}{3}, \quad B = -\frac{1}{12}, \quad C = -\frac{1}{2}$

$D = \frac{2}{6} - \frac{1}{144} = \frac{1}{3} - \frac{1}{144} > 0$

f-ja z ima ekstrem

$A < 0$ f-ja ima maksimum

$$Z_{max}(21,20) = 21\cdot 10 + (47 - 41)(7 + 5) = 210 + 6\cdot 12 = 210 + 72 = 282$$

$Z_{max}(21,20) = 282$ traženi ekstrem f-je

◯ Naði ekstreme f-je $z = x^4 + y^4 - 2x^2$.

Rj.

$$\frac{\partial z}{\partial x} = 4x^3 - 4x$$

$$\frac{\partial z}{\partial Y} = 4y^3$$

$$\begin{array}{ll} 4x^3 - 4x = 0 & /:4 \\ 4Y^3 = 0 & /:4 \\ \hline x^3 - x = 0 & \\ Y^3 = 0 & \\ \hline \end{array} \qquad \begin{array}{l} x(x^2 - 1) = 0 \\ \cdot\, Y^3 = 0 \\ \hline x(x-1)(x+1) = 0 \\ Y^3 = 0 \\ \hline \end{array}$$

$$Y = 0 \ \wedge \ (x_1 = 0,\ x_2 = 1,\ x_3 = -1)$$

Stacionarne tačke f-je su $M_1(-1,0)$, $M_2(0,0)$ i $M_3(1,0)$.

$$\frac{\partial^2 z}{\partial x^2} = 12x^2 - 4$$

$$\frac{\partial^2 z}{\partial x \partial Y} = 0$$

$$\frac{\partial^2 z}{\partial Y^2} = 12y^2$$

za $M_1(-1,0)$, $A = 8$, $B = 0$, $C = 0$

$D = 0$ ispitujemo ponašanje f-je u okolini tačke $M_1(-1,0)$

paskalov trougao

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$\Delta z = z(-1+\varepsilon, 0+\omega) - z(-1,0) =$$

$$= (-1+\varepsilon)^4 + \omega^4 - 2(-1+\varepsilon)^2 - \left[(-1)^4 + 0^4 - 2(-1)^2\right]$$

$$= 1 - 4\varepsilon + 6\varepsilon^2 - 4\varepsilon^3 + \varepsilon^4 + \omega^4 - 2(1 - 2\varepsilon + \varepsilon^2) - (1 - 2)$$

$$= \underline{\underline{1}} \underline{-4\varepsilon} + \underline{\underline{6\varepsilon^2}} - 4\varepsilon^3 + \varepsilon^4 + \omega^4 \underline{\underline{-2}} \underline{+4\varepsilon} \underline{\underline{-2\varepsilon^2}} + \underline{\underline{1}}$$

$$= \varepsilon^4 - 4\varepsilon^3 + 4\varepsilon^2 + \omega^4 = \varepsilon^2(\varepsilon^2 - 4\varepsilon + 4) + \omega^4$$

$$= \varepsilon^2(\varepsilon - 2)^2 + \omega^4 \quad \geq 0 \quad \text{za } \forall\varepsilon \text{ i } \forall\omega$$

f-ja ima minimum u tački $M_1(-1,0)$, $Z_{min} = -1$

za $M_2(0,0)$, $A = -4$, $B = 0$, $C = 0$, $D = AC - B^2 = 0$

ispitujemo ponašanje f-je u okolini tačke

$$\Delta z = z(0+\varepsilon, 0+\omega) - z(0,0) = \varepsilon^4 + \omega^4 - 2\varepsilon^2 = \varepsilon^2(\varepsilon^2 - 2) + \omega^4$$

$\varepsilon = 0: \quad \Delta z = \omega^4$

$\omega = 0: \quad \Delta z = \varepsilon^2(\varepsilon^2 - 2) \Rightarrow \Delta z < 0$ za $\varepsilon^2 < 2$

$\Delta z > 0$ za $\varepsilon^2 > 2$

Priraštaj f-je je promjenjivog znaka pa f-ja nema ekstrem u tački M_2

za $M_3(1,0)$, $A = 8$, $B = 0$, $C = 0$, $D = AC - B^2 = 0$ ispitujemo ponašanje f-je u okolini tačke

$$\Delta z = z(1+\varepsilon, 0+\omega) - z(1,0) = (1+\varepsilon)^4 + \omega^4 - 2(1+\varepsilon)^2 - (1-2)$$

$$= \underline{1} + \underline{4\varepsilon} + \underline{\underline{6\varepsilon^2}} + \underline{\underline{4\varepsilon^3}} + \underline{\varepsilon^4} + \omega^4 \underline{-2} \underline{-4\varepsilon} \underline{\underline{-2\varepsilon^2}} \underline{+1} = \varepsilon^4 + 4\varepsilon^3 + 4\varepsilon^2 + \omega^4$$

$$\Delta z = \varepsilon^2(\varepsilon + 2)^2 + \omega^4 \geq 0 \quad \forall\varepsilon \text{ i } \forall\omega$$

f-ja z u tački M_3 ima min

$Z_{min} = -1$

(#) Naći ekstreme f-je $z=(2x^2+3y^2)e^{-(x^2+y^2)}$.

Rj.

$$\frac{\partial z}{\partial x}=4x\cdot e^{-x^2-y^2}+(2x^2+3y^2)e^{-x^2-y^2}\cdot(-2x)=(4x-4x^3-6xy^2)e^{-x^2-y^2}$$

$$\frac{\partial z}{\partial y}=6ye^{-x^2-y^2}+(2x^2+3y^2)e^{-x^2-y^2}\cdot(-2y)=(6y-4x^2y-6y^3)e^{-x^2-y^2}$$

$$2x(2-2x^2-3y^2)e^{-x^2-y^2}=0$$
$$2y(3-2x^2-3y^2)e^{-x^2-y^2}=0$$

$e^{-x^2-y^2}\neq 0\ \forall(x,y\in\mathbb{R})$

$x=0$ i $y=0$, $M_1(0,0)$

ili

$x=0$ i $3-2x^2-3y^2=0$

$3y^2=3$

$y^2=1$

$y_{1,2}=\pm1$

$M_2(0,-1)$

$M_3(0,1)$

ili $2-2x^2-3y^2=0$ i $y=0$

$2x^2=2$

$x^2=1$

$x_{1,2}=\pm1$

$M_4(-1,0)$

$M_5(1,0)$

ili

$$\begin{array}{r} 2-2x^2-3y^2=0\\ -\ 3-2x^2-3y^2=0\\ \hline -1=0 \end{array}$$

sistem nema rješenja

Stacionarne tačke su M_1, M_2, M_3, M_4 i M_5.

$$\frac{\partial^2 z}{\partial x^2}=(4-\underline{12x^2}-6y^2)e^{-x^2-y^2}+(\underline{4x}-\underline{4x^3}-\underline{6xy^2})e^{-x^2-y^2}(-2x)=(8x^4+12x^2y^2-20x^2-6y^2+4)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial x\partial y}=(-12xy)e^{-x^2-y^2}+(\underline{4x}-\underline{4x^3}-6xy^2)e^{-x^2-y^2}(-2y)=(-20xy+8x^3y+12xy^3)e^{-x^2-y^2}$$

$$\frac{\partial^2 z}{\partial y^2}=(6-4x^2-\underline{18y^2})e^{-x^2-y^2}+(\underline{6y}-\underline{4x^2y}-\underline{6y^3})e^{-x^2-y^2}\cdot(-2y)=(-30y^2+12y^4+8x^2y^2-4x^2+6)e^{-x^2-y^2}$$

za $M_1(0,0)$, $A=4$, $B=0$, $C=6$, $D=AC-B^2=24>0$ ima ekstrem

$A>0$ ima minimum, $z_{min}(0,0)=0$

za $M_2(0,-1)$, $A=-2e^{-1}$, $B=0$, $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum, $z_{max}(0,-1)=3e^{-1}$

za $M_3(0,1)$, $A=-2e^{-1}$, $B=0$ $C=-12e^{-1}$, $D=AC-B^2=24e^{-2}>0$ ima ekstrem

$A<0$ ima maksimum $z_{max}(0,1)=3e^{-1}$

za $M_4(-1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$, $D=AC-B^2=-16e^{-2}<0$

f-ja u tački $M_4(-1,0)$ nema ekstrem

za $M_5(1,0)$, $A=-8e^{-1}$, $B=0$, $C=2e^{-1}$,

f-ja u tački $M_5(1,0)$ nema ekstrem

(#) Naći stacionarne tačke f-je $z=xy\ln(x^2+y^2)$.

Rj.

$$\frac{\partial z}{\partial x}=y\ln(x^2+y^2)+xy\cdot\frac{1}{x^2+y^2}\cdot 2x=y\ln(x^2+y^2)+\frac{2x^2y}{x^2+y^2}$$

$$\frac{\partial z}{\partial y}=x\ln(x^2+y^2)+xy\cdot\frac{1}{x^2+y^2}\cdot 2y=x\ln(x^2+y^2)+\frac{2xy^2}{x^2+y^2}$$

$$y\ln(x^2+y^2)+\frac{2x^2y}{x^2+y^2}=0$$
$$x\ln(x^2+y^2)+\frac{2xy^2}{x^2+y^2}=0$$

$$y\left(\ln(x^2+y^2)+\frac{2x^2}{x^2+y^2}\right)=0$$
$$x\left(\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}\right)=0$$

$y=0$ ili $\ln(x^2+y^2)+\dfrac{2x^2}{x^2+y^2}=0$

$x=0$ ili $\ln(x^2+y^2)+\dfrac{2y^2}{x^2+y^2}=0$

$y=0$ i $x=0$ $M_1(0,0)$

za M_1 f-ja nije definisana

ili

$y=0$ i $\ln(x^2+y^2)+\dfrac{2y^2}{x^2+y^2}=0$

$\ln x^2=0$

$x^2=1$

$x_1=-1$, $x_2=1$

$M_2(-1,0)$, $M_3(1,0)$

ili

$\ln(x^2+y^2)+\dfrac{2x^2}{x^2+y^2}=0$ i $x=0$

$\ln y^2=0$

$y_1=-1$ $y_2=1$

$M_4(0,-1)$

$M_5(0,1)$

ili

$$\ln(x^2+y^2)+\frac{2x^2}{x^2+y^2}=0 \quad (1)$$
$$\ln(x^2+y^2)+\frac{2y^2}{x^2+y^2}=0 \quad (2)$$

(1)−(2): $\dfrac{2x^2}{x^2+y^2}-\dfrac{2y^2}{x^2+y^2}=0$

$2x^2-2y^2=0$

$2(x^2-y^2)=0$

$2(x-y)(x+y)=0$

$x=y$ ili $x=-y$.

za $x=y$: $\ln(2x^2)+1=0$

$\ln(2x^2)=-1$

$e^{-1}=2x^2$

$x^2=\dfrac{1}{2e}$

$x_{1,2}=\pm\dfrac{1}{\sqrt{2e}}$

$M_6\left(-\dfrac{1}{\sqrt{2e}},-\dfrac{1}{\sqrt{2e}}\right)$, $M_7\left(\dfrac{1}{\sqrt{2e}},\dfrac{1}{\sqrt{2e}}\right)$

za $y=-x$: $\ln(2x^2)+1=0$

$x_{1,2}=\pm\dfrac{1}{\sqrt{2e}}$

$M_8\left(-\dfrac{1}{\sqrt{2e}},\dfrac{1}{\sqrt{2e}}\right)$

$M_9\left(\dfrac{1}{\sqrt{2e}},-\dfrac{1}{\sqrt{2e}}\right)$

Stacionarne tačke su: $M_2, M_3, M_4, M_5, M_6, M_7, M_8$ i M_9.

Uslovni ekstremi f-ja dviju promjenjivih

Ako trebamo naći ekstrem f-je $z=f(x,y)$ tako da x i y zadovoljavaju neki uslov $g(x,y)=0$ tada tražimo ekstrem Lagranžove f-je $F(x,y;\lambda)=f(x,y)+\lambda g(x,y)$.

$$\underline{\begin{aligned} \frac{\partial F}{\partial x}&=0\\ \frac{\partial F}{\partial y}&=0\\ \frac{\partial F}{\partial z}&=0 \end{aligned}}$$

SISTEM

rješavanjem sistema dobićemo neke stacionarne tačke i dalji proces se nastavlja kao kod traženja ekstrema f-je dvije promjenjive

II način: neka je $M(p_1,p_2)$ neka stacionarna tačka

$$d^2F(p_1,p_2)=F''_{xx}(p_1,p_2)\,dx^2+2F''_{xy}(p_1,p_2)\,dx\,dy+F''_{yy}(p_1,p_2)\,dy^2$$

$$d^2F(p_1,p_2)>0 \quad\Rightarrow\quad z_{min}(p_1,p_2)$$

$$d^2F(p_1,p_2)<0 \quad\Rightarrow\quad z_{max}(p_1,p_2)$$

Ako se desi slučaj da imamo više uslova, onda uvodimo više parametara $(\lambda,\mu,\dots)$.

① Naći ekstreme f-je $z=6-4x-3y$ uz uslov $x^2+y^2=1$.

Rj. $F(x,y)=6-4x-3y+\lambda(x^2+y^2-1)\neq 0$

$$\frac{\partial F}{\partial x}=-4+2\lambda x$$

$$\frac{\partial F}{\partial y}=-3+2\lambda y$$

$$\frac{\partial F}{\partial \lambda}=x^2+y^2-1$$

$$\underline{\begin{aligned} 2\lambda x-4&=0\\ 2\lambda y-3&=0\\ x^2+y^2-1&=0 \end{aligned}}$$

$$\underline{\begin{aligned} 2\lambda x&=4\\ 2\lambda y&=3\\ x^2+y^2&=1 \end{aligned}}$$

$$x=\frac{2}{\lambda}$$

$$\underline{\begin{aligned} y&=\frac{3}{2\lambda}\\ x^2+y^2&=1 \end{aligned}}$$

$$\frac{4}{\lambda^2}+\frac{9}{4\lambda^2}=1$$

$$\frac{25}{4\lambda^2}=1$$

$$4\lambda^2=25$$

$$\lambda_{1,2}=\pm\frac{5}{2}$$

$$\lambda_1=-\frac{5}{2}\Rightarrow x_1=-\frac{4}{5}\ ;\ y_1=\frac{3}{2\cdot(-\frac{5}{2})}=-\frac{3}{5}$$

$$\lambda_2=\frac{5}{2}\Rightarrow x_2=\frac{2}{\frac{5}{2}}=\frac{4}{5}\ ;\ y_2=\frac{3}{2\cdot\frac{5}{2}}=\frac{3}{5}$$

Stacionarne tačke su $M(-\frac{4}{5},-\frac{3}{5})$ za $\lambda=-\frac{5}{2}$ i $N(\frac{4}{5},\frac{3}{5})$ za $\lambda=\frac{5}{2}$.

$$\frac{\partial^2 F}{\partial x^2}=2\lambda$$

$$\frac{\partial^2 F}{\partial x\partial y}=0$$

$$\frac{\partial^2 F}{\partial y^2}=2\lambda$$

za $M(-\frac{4}{5},-\frac{3}{5})$, $\lambda=-\frac{5}{2}$

$A=-5$, $B=0$, $C=-5$, $D=AC-B^2=25>0$

f-ja ima ekstrem, $A<0$ f-ja ima maksimum

$$z_{max}\left(-\frac{4}{5},-\frac{3}{5}\right)=6-4\left(-\frac{4}{5}\right)-3\left(-\frac{3}{5}\right)=\frac{30+16+9}{5}=\frac{55}{5}=11$$

za $N(\frac{4}{5},\frac{3}{5})$, $\lambda=\frac{5}{2}$, $A=5$, $B=0$, $C=5$, $D=AC-B^2=25>0$

f-ja ima ekstrem u tački N, $A>0$ f-ja ima minimum

$$z_{min}\left(\frac{4}{5},\frac{3}{5}\right)=6-4\cdot\frac{4}{5}-3\cdot\frac{3}{5}=\frac{30-16-9}{5}=\frac{5}{5}=1$$

② Naći uslovne ekstreme f-je $z=y+2x+3$ uz uslov $x^2-6x+y+5=0$.

Rj. $F(x,y)=2x+y+3+\lambda(x^2-6x+y+5)$

$$\frac{\partial F}{\partial x}=2+2\lambda x-6\lambda$$

$$\frac{\partial F}{\partial y}=1+\lambda$$

$$\frac{\partial F}{\partial \lambda}=x^2-6x+y+5$$

$$\underline{\begin{aligned} 2\lambda x-6\lambda+2&=0 \quad /:2\\ \lambda+1&=0\\ x^2-6x+y+5&=0 \end{aligned}}$$

$$\underline{\begin{aligned} \lambda x&=3\lambda-1\\ \lambda&=-1\\ x^2-6x+y+5&=0 \end{aligned}}$$

$$-x=-3-1$$

$$x=4$$

$$x^2-6x+y+5=0$$

$$16-24+y+5=0$$

$$y=3$$

Tačka $M(4,3)$ je stacionarna tačka, za $\lambda=-1$

$$\frac{\partial^2 F}{\partial x^2}=2\lambda$$

$$\frac{\partial^2 F}{\partial x\partial y}=0$$

$$\frac{\partial^2 F}{\partial y^2}=0$$

$M(4,3)$, $\lambda=-1$

$A=-2$, $B=0$, $C=0$ $\Rightarrow$ $D=AC-B^2=0$

$$d^2F=\frac{\partial^2 F}{\partial x^2}dx^2+\frac{\partial^2 F}{\partial x\partial y}\cancel{dxdy}+\frac{\partial^2 F}{\partial y^2}dy^2$$

$$d^2F=2\lambda\,dx^2\quad\Rightarrow\quad d^2F=-2\,dx^2<0$$

U tački $M(4,3)$ f-ja ima maksimum, $z_{max}(4,3)=3+8+3=14$

③ Odrediti ekstreme f-je $z=x^2+y^2$ uz uslov $\frac{x}{2}+\frac{y}{3}=1$.

Rj. $z_{min}\left(\frac{18}{13},\frac{12}{13}\right)=\frac{36}{13}$, $\lambda=-\frac{72}{13}$

④ Nađi uslovne ekstreme f-je $z=\ln(x+y)$, ako je $x^2+2y^2=4$.

Rj. $z_{max}\left(2\sqrt{\frac{2}{3}},\sqrt{\frac{2}{3}}\right)=\ln\left(3\sqrt{\frac{2}{3}}\right)$, $\lambda=-\frac{1}{8}$.

(#) Naći uslovne ekstreme f-je $z = 2x^4 + 8y^4 + 24$ ako je $8x + 4y = 1$.

Rj. $F(x, y, \lambda) = 2x^4 + 8y^4 + 24 + \lambda(8x + 4y - 1)$

$$\frac{\partial F}{\partial x} = 8x^3 + 8\lambda$$

$$\frac{\partial F}{\partial y} = 32y^3 + 4\lambda$$

$$\frac{\partial F}{\partial \lambda} = 8x + 4y - 1$$

$$8x^3 + 8\lambda = 0 \quad /:8$$

$$32y^3 + 4\lambda = 0 \quad /:4$$

$$x^3 + \lambda = 0$$

$$- \quad 8y^3 + \lambda = 0$$

$$x^3 - 8y^2 = 0$$

$$x^3 = 8y^3$$

$$x = 2y$$

$$8x + 4y - 1 = 0$$

$$8 \cdot 2y + 4y - 1 = 0$$

$$20y = 1$$

$$y = \frac{1}{20}$$

$$x = 2 \cdot \frac{1}{20} = \frac{1}{10}$$

$M_1\left(\frac{1}{10}, \frac{1}{20}\right)$ je stacionarna tačka

$$\frac{\partial^2 F}{\partial x^2} = 24x^2$$

$$\frac{\partial^2 F}{\partial x \partial y} = 0$$

$$\frac{\partial^2 F}{\partial y^2} = 96y^2$$

$$D = AC - B^2$$

$M_1\left(\frac{1}{10}, \frac{1}{20}\right)$

$$A = 24 \cdot \frac{1}{100} = \frac{24}{100} = \frac{12}{50} = \frac{6}{25}$$

$$B = 0$$

$$C = \overset{\overset{24}{\cancel{48}}}{\cancel{96}} \cdot \frac{1}{\underset{10}{\cancel{20}} \cdot \underset{10}{\cancel{20}}} = \frac{24}{100} = \frac{12}{50} = \frac{6}{25}$$

$D = \left(\frac{6}{25}\right)^2 > 0$ f-ja ima ekstrem

$A > 0$ f-ja ima minimum

$$Z_{min}\left(\frac{1}{10}, \frac{1}{20}\right) = 2 \cdot \frac{1}{10^4} + 8 \cdot \frac{1}{20^4} + 24 = \frac{2}{10\,000} + \frac{\overset{\overset{\overset{1}{\cancel{2}}}{\cancel{4}}}{\cancel{8}}}{\underset{10}{\cancel{20}} \cdot \underset{10}{\cancel{20}} \cdot \underset{10}{\cancel{20}} \cdot 20} + 24$$

$$= \frac{2}{10\,000} + \frac{1}{20\,000} + \frac{24\,000}{1\,000} = \frac{4 + 1 + 480\,000}{20\,000} =$$

$$= \frac{480\,005 \;:5}{20\,000 \;:5} = \frac{96\,001}{4\,000}$$

$Z_{min} = \frac{96\,001}{4\,000}$ je minimum f-je u tački $M\left(\frac{1}{10}, \frac{1}{20}\right)$

(#) Naći uslovne ekstreme f-je $z = (x - y)^4 + 1$ ako je $x^2 + y^2 = 18$.

Rj. $F(x, y, \lambda) = (x - y)^4 + 1 + \lambda(x^2 + y^2 - 18)$

$$\frac{\partial F}{\partial x} = 4(x - y)^3 + 2\lambda x$$

$$\frac{\partial F}{\partial y} = 4(x - y)^3 \cdot (-1) + 2\lambda y$$

$$\frac{\partial F}{\partial \lambda} = x^2 + y^2 - 18$$

$$4(x - y)^3 + 2\lambda x = 0 \quad \ldots(1)$$

$$-4(x - y)^3 + 2\lambda y = 0 \quad \ldots(2)$$

$$x^2 + y^2 - 18 = 0 \quad \ldots(3)$$

$(1)\ (2):\ 2\lambda x + 2\lambda y = 0 \quad /:2$

$$\lambda(x + y) = 0$$

$\lambda = 0$ ili $x + y = 0$

a) $x + y = 0$

$$x = -y$$

$$(-y)^2 + y^2 = 18$$

$$2y^2 = 18$$

$$y_{1,2} = \pm 3$$

$y_1 = -3 \Rightarrow x_1 = 3$

$y_2 = 3 \Rightarrow x_2 = -3$

$M_1(3, -3),\ M_2(-3, 3)$

za M_1 (1) $\Rightarrow 6\lambda = -4 \cdot 6^3$

za M_2 (1) $\Rightarrow -6\lambda = -4 \cdot (-6)^3$

$\Rightarrow \lambda = -144$

b) $\lambda = 0$

(1) $\Rightarrow 4(x - y)^3 = 0$

$$x = y$$

$$2y^2 = 18$$

$$y_{3,4} = \pm 3$$

$M_3(-3, -3)$

$M_4(3, 3)$

$\lambda = 0$

Stacionarne tačke su $M_1(3, -3)$ $M_2(-3, 3)$ za $\lambda = -144$ i $M_3(-3, -3)$ i $M_4(3, 3)$ za $\lambda = 0$.

$$\frac{\partial^2 F}{\partial x^2} = 12(x - y)^2 + 2\lambda$$

$$\frac{\partial F}{\partial x \partial y} = -12(x - y)^2$$

$$\frac{\partial F}{\partial y^2} = 12(x - y)^2 + 2\lambda$$

$$D = AC - B^2$$

$M_1(3, -3),\ \lambda = -144$

$$A = 12 \cdot 36 - 2 \cdot 144 = 144$$

$$B = -12 \cdot 36 = -432$$

$$C = 144$$

$$D = 20\,736 - 186\,624$$

f-ja u tački M_1 nema ekstrema

$M_2(-3, 3),\ \lambda = -144$

$$\left.\begin{array}{l} A = 144 \\ B = 432 \\ C = 144 \end{array}\right\} D < 0$$

f-ja u tački M_2 nema ekstrem

$M_3(-3, -3),\ \lambda = 0$

$A = 0,\ B = 0,\ C = 0 \Rightarrow D = 0$ potrebno je ispitati f-ju u okolini tačke $M_3(-3, -3)$

$$\Delta z(M_3) = z(-3 + \varepsilon, -3 + \omega) - z(-3, -3) = (-3 + \varepsilon + 3 - \omega)^4 + 1 - 1 = (\varepsilon - \omega)^4 > 0 \quad \forall \varepsilon \text{ i } \forall \omega$$

Priraštaj f-je u okolini tačke M_3 je pozitivan pa f-ja u M_3 ima minimum, $Z_{min}(-3, -3) = 1$

$M_4(3, 3),\ \lambda = 0$

$A = 0,\ B = 0,\ C = 0 \Rightarrow D = 0$ potrebno je ispitati f-ju u okolini tačke $M_4(3, 3)$

$$\Delta z(M_4) = z(3 + \varepsilon, 3 + \omega) - z(3, 3) = (3 + \varepsilon - 3 - \omega)^4 + 1 - 1 = (\varepsilon - \omega)^4 > 0 \quad \forall \varepsilon \text{ i } \forall \omega$$

Priraštej f-je u okolini tačke M_4 je pozitivan pa f-ja u M_4 ima minimum, $Z_{min}(3, 3) = 1$.

(#) Nađi uslovne ekstreme f-je $z=2x+4y$ ako je
$\frac{2}{x}+\frac{4}{y}=3$.

Rj. Formirajmo Lagranžovu f-ju $F(x,y,\lambda)=2x+4y+\lambda\left(\frac{2}{x}+\frac{4}{y}-3\right)$.

$$\frac{\partial F}{\partial x}=2+2\lambda\cdot\frac{(-1)}{x^2}$$

$$\left(\frac{1}{x}\right)'=(x^{-1})'=(-1)(x^{-2})$$

$$(x^{-2})'=(-2)x^{-3}=\frac{-2}{x^3}$$

$$\frac{\partial F}{\partial y}=4+4\lambda\cdot\frac{(-1)}{y^2}$$

$$\frac{\partial F}{\partial \lambda}=\frac{2}{x}+\frac{4}{y}-3$$

Formirajmo sistem

$$4-\frac{4\lambda}{y^2}=0 \quad /:4$$
$$2-\frac{2\lambda}{x^2}=0 \quad /:2$$
$$\frac{2}{x}+\frac{4}{y}=3$$

$$1-\frac{\lambda}{x^2}=0 \qquad 1=\frac{\lambda}{x^2} \quad (1)$$
$$1-\frac{\lambda}{y^2}=0 \qquad 1=\frac{\lambda}{y^2} \quad (2)$$
$$\frac{2}{x}+\frac{4}{y}=3 \qquad \frac{2}{x}+\frac{4}{y}=3 \quad (3)$$

(1) i (2) $\Rightarrow \frac{\lambda}{x^2}=\frac{\lambda}{y^2} \Rightarrow x^2=y^2$

tj. $x=\pm y$

Za $x=y$ iz (3) $\frac{2}{x}+\frac{4}{x}=3$

$\frac{6}{x}=3 \Rightarrow x=2 \Rightarrow y=2$

Za $x=-y$ iz (3)

$\frac{2}{x}-\frac{4}{x}=3 \Rightarrow -\frac{2}{x}=3$

$3x=-2 \Rightarrow x=-\frac{2}{3}$

$\Rightarrow y=\frac{2}{3}$

Za $M_1(2,2) \Rightarrow 2-2\lambda\cdot\frac{1}{4}=0$

$\lambda=4$

Za $M_2(-\frac{2}{3},\frac{2}{3}) \Rightarrow 2-2\lambda\cdot\frac{9}{4}=0 \Rightarrow \lambda=\frac{4}{9}$

Stacionarne tačke su $M_1(2,2)$ za $\lambda=4$ i $M_2(-\frac{2}{3},\frac{2}{3})$ za $\lambda=\frac{4}{9}$.

$$\frac{\partial^2 F}{\partial x^2}=\frac{4\lambda}{x^3}$$

$$\frac{\partial^2 F}{\partial x\partial y}=0$$

$$\frac{\partial^2 F}{\partial y^2}=\frac{8\lambda}{y^3}$$

Za $M_1(2,2)$, $\lambda=4$

$A=\frac{16}{8}=2$, $B=0$, $C=\frac{32}{8}=4$, $D=AC-B^2=8>0$ f-ja ima ekstrem

$A>0 \Rightarrow$ f-ja ima minimum

$z_{min}(2,2)=4+8=12$

Za $M_2(-\frac{2}{3},\frac{2}{3})$, $\lambda=\frac{4}{9}$, $A=\frac{\frac{16}{9}}{-\frac{8}{27}}=-\frac{16\cdot 27}{8\cdot 9}=-2\cdot 3=-6$

$B=0$, $C=\frac{\frac{32}{9}}{\frac{8}{27}}=\frac{32\cdot 27}{8\cdot 9}=4\cdot 3=12$, $D=AC-B^2=-72<0 \Rightarrow$

$\Rightarrow$ f-ja u tački M_2 nema ekstremnu vrijednost

(#) Nađi uslovne ekstreme f-je $z=xy$ ako je
$x^2+y^2=2ax$, $a>0$.

Rj. Posmatramo f-ju $F(x,y,\lambda)=xy+\lambda(x^2+y^2-2ax)$

$$\frac{\partial F}{\partial x}=y+2\lambda x-2a\lambda$$

$$\frac{\partial F}{\partial y}=x+2\lambda y$$

$$\frac{\partial F}{\partial \lambda}=x^2+y^2-2ax$$

$$y+2\lambda x-2a\lambda=0$$
$$x+2\lambda y=0$$
$$x^2+y^2-2a\cdot x=0$$

(1) $y+2\lambda(x-a)=0 \Rightarrow x-a=\frac{-y}{2\lambda}$ ...(*)

(2) $x=-2\lambda y$

(3) $x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$

(*) u (3):

$$\frac{y^2}{4\lambda^2}+y^2=a^2$$
$$y^2\left(\frac{1}{4\lambda^2}+1\right)=a^2$$
$$y^2\left(\frac{1+4\lambda^2}{4\lambda^2}\right)=a^2$$
$$y^2=\frac{4a^2\lambda^2}{1+4\lambda^2}$$
$$y=\frac{2a\lambda}{\pm\sqrt{1+4\lambda^2}}$$

(2) u (1): $y+2\lambda(-2\lambda y-a)=0$

(3): $(x-a)^2+y^2=a^2$

$$y-4\lambda^2y-2a\lambda=0$$
$$y(1-4\lambda^2)=2a\lambda$$
$$y=\frac{2a\lambda}{1-4\lambda^2}$$

$$y=\frac{2a\lambda}{1-4\lambda^2}=\frac{2a\lambda}{\pm\sqrt{1+4\lambda^2}} \Rightarrow 1-4\lambda^2=\pm\sqrt{1+4\lambda^2}$$

$$(1-4\lambda^2)^2=1+4\lambda^2$$
$$16\lambda^4-8\lambda^2+1=1+4\lambda^2$$
$$16\lambda^4-12\lambda^2=0$$
$$\lambda^2(16\lambda^2-12)=0$$

$\lambda_1=0$

$\lambda_{2,3}=\pm\sqrt{\frac{12}{16}}=\pm\sqrt{\frac{3}{4}}$

$=\pm\frac{\sqrt{3}}{2}$

$\lambda_1=0$: $y=0$
$x=0$

$\lambda_2=\frac{\sqrt{3}}{2}$:

$$y+\sqrt{3}x-a\sqrt{3}=0$$
$$x+y\sqrt{3}=0 \quad /\cdot\sqrt{3}$$
$$x^2+y^2-2ax=0$$

$$\sqrt{3}x+y=a\sqrt{3}$$
$$-\ \sqrt{3}x+3y=0$$
$$-2y=a\sqrt{3}$$

$y=\frac{-a}{2}\sqrt{3}$ $\quad x=\frac{3}{2}a$

$\lambda_3=-\frac{\sqrt{3}}{2}$:

$$y-x\sqrt{3}+a\sqrt{3}=0$$
$$x-y\sqrt{3}=0 \quad /\cdot\sqrt{3}$$
$$x^2+y^2-2ax=0$$

$$-x\sqrt{3}+y=-a\sqrt{3}$$
$$+\ x\sqrt{3}-3y=0$$
$$-2y=-a\sqrt{3}$$

$y=\frac{\sqrt{3}}{2}a \Rightarrow x=\frac{3}{2}a$

Stacionarne tačke su $M_1(0,0)$ za $\lambda=0$, $M_2(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a)$ za $\lambda=\frac{\sqrt{3}}{2}$; $M_3(\frac{3}{2}a, \frac{\sqrt{3}}{2}a)$ za $\lambda=-\frac{\sqrt{3}}{2}$.

$$\frac{\partial^2 F}{\partial x^2}=2\lambda$$

$$\frac{\partial^2 F}{\partial x \partial y}=1$$

$$\frac{\partial^2 F}{\partial y^2}=2\lambda$$

$M_1(0,0)$, $\lambda=0$

$D=AC-B^2=-1<0 \Rightarrow$ f-ja u tački $M_1(0,0)$ nema ekstrem

$M_2(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a)$, $\lambda=\frac{\sqrt{3}}{2}$

$D=AC-B^2=3-1=2>0 \Rightarrow$ f-ja u tački M_2 ima ekstrem

$A=\sqrt{3}>0 \Rightarrow$ f-ja ima minimum

$$Z_{min}\left(\frac{3}{2}a, -\frac{\sqrt{3}}{2}a\right)=-\frac{3\sqrt{3}}{4}a^2$$

$M_3(\frac{3}{2}a, \frac{\sqrt{3}}{2}a)$ za $\lambda=-\frac{\sqrt{3}}{2}$

$D=AC-B^2=3-1>0 \Rightarrow$ f-ja ima ekstrem

$A=-\sqrt{3}<0 \Rightarrow$ f-ja u tački M_3 ima maksimum

$$Z_{max}\left(\frac{3}{2}a, \frac{\sqrt{3}}{2}a\right)=\frac{3\sqrt{3}}{4}a^2$$

Zadaci za vježbu

Lokalne ekstremne vrednosti

U zadacima 3259 — 3267 naći stacionarne tačke datih funkcija.

3259. $z=2x^3+xy^2+5x^2+y^2$. **3260.** $z=e^{2x}(x+y^2+2y)$.

3261. $z=xy(a-x-y)$. **3262.** $z=(2ax-x^2)(2by-y^2)$.

3263. $z=\sin x+\sin y+\cos(x+y)\left(0<x<\frac{\pi}{4},\ 0<y<\frac{\pi}{4}\right)$.

3264. $z=\frac{a+bx+cy}{\sqrt{1+x^2+y^2}}$ **3265.** $z=y\sqrt{1+x}+x\sqrt{1+y}$.

3266. $u=2x^2+y^2+2z-xy-xz$.

3267. $u=3\ln x+2\ln y+5\ln z+\ln(22-z-y-z)$.

3268. **Na sl. 60 predstavljene su nivoske linije funkcije $z=fx, y)$. Kakve osobenosti pokazuje ova funkcija u Tačkama A, B, C, D, i na pravoj EF?**

3269. **Funkcija z definisana je implicitno jednačinom**

$$2x^2+2y^2+z^2+8xz-z+8=0.$$

Naći njene stacionarne tačke.

3270. **Funkcija z definisana je implicitno jednačinom**

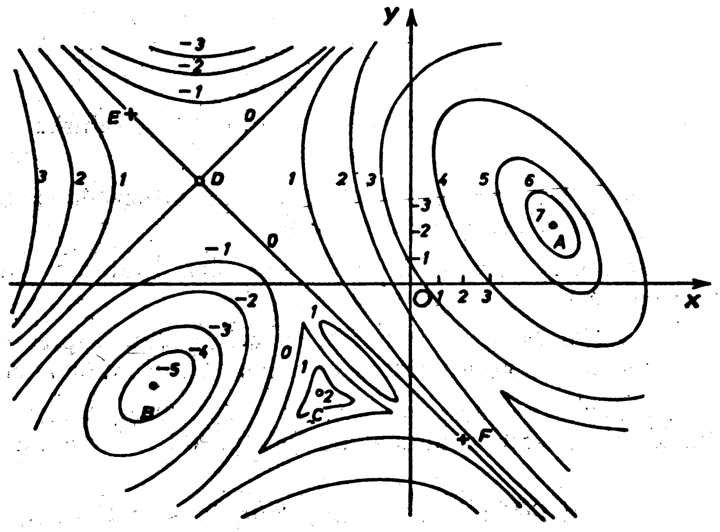
$$5x^2+5y^2+5z^2-2xy-2xz-2yz-72=0.$$

Naći njene stacionarne tačke.

3271*. **Naći. tačke ekstremuma funkcije**

$$z=2xy-3x^2-2y^2+10.$$

3272. **Naći tačke ekstremuma funkcije $z=4(x-y)-x^2-y^2$.**



Sl. 60

3273. Naći tačke ekstremuma funkcije $z=x^2+xy+y^2+x-y+1$.

3274. Uveriti se da funkcija $z=x^2+xy+y^2+\frac{a^3}{x}+\frac{a^3}{y}$ ima minimum u tački $x=y=\frac{a}{\sqrt[3]{3}}$.

3275. Uveriti se da za $x=\sqrt{2}$, $y=\sqrt{2}$ i za $x=-\sqrt{2}$, $y=-\sqrt{2}$ funkcija $z=x^4+y^4-2x^2-4xy-2y^2$ ima minimum.

3276. Uveriti se da za $x=5$, $y=6$ funkcija $z=x^3+y^2-6xy-39x+18y+20$ ima minimum.

3277. Naći stacionarne tačke funkcije $z=x^3y^2(12-x-y)$, koje zadovoljavaju uslov $x>0$, $y>0$ i ispitati njihov karakter.

3278. Naći stacionarne tačke funkcije $z=x^3+y^3-3xy$ i ispitati njihov karakter.

Ekstremne vrednosti u datoj oblasti

3279. Naći najveću *i* najmanju vrednost funkcije $z=x^2-y^2$ u krugu $x^2+y^2<4$.

3280. Naći najveću i najmanju vrednost funkcije $z=x^2+2xy-4x+8y$ u pravougaoniku $0<x<1$, $0<y<2$.

3281. Naći najveću vrednost funkcije $z=x^2y(4-x-y)$ u trouglu koji obrazuju prave $x=0$, $y=0$, $x+y=6$.

3282. Naći najveću i najmanju vrednost funkcije $z=e^{-x^2-y^2}(2x^2+3y^2)$ u krugu $x^2+y^2<4$.

3283. Naći najveću i najmanju vrednost funkcije

$$z=\sin x+\sin y+\sin(x+y)$$

u pravougaoniku $0<x<\frac{\pi}{2}$; $0<y<\frac{\pi}{2}$.

3284. Pozitivan broj a razložiti na tri proizvoljna sabirka tako da njihov proizvod bude minimalan.

3285. Pozitivan broj a predstaviti u obliku proizvoda četiri pozitivna množitelja tako da njihov zbir bude minimalan.

3286. U ravni Oxy naći tačku za koju je zbir kvadrata odstojanja od pravih $x=0$, $y=0$, $x+2y-16=0$ minimalan.

3287. Kroz tačku (a, b, c) postaviti ravan tako da zapremina tetraedra koji ta ravan obrazuje sa koordinatnim ravnima, bude minimalna.

3288. Date su tačke $A_1(x_1, y_1, z_1), \ldots, A_n(x_n, y_n, z_n)$; u ravni Oxy naći tačku za koju će zbir kvadrata odstojanja od svih datih tačaka biti minimalan.

3289. Date su tri tačke $A(0, 0, 12)$, $B(0, 0, 4)$ i $C(8, 0, 8)$; u ravni Oxy naći tačku D tako da poluprečnik sfere koja prolazi kroz tačke $ABCD$ bude minimalan.

3290. U loptu prečnika $2R$ upisati pravougli paralelepiped maksimalne zapremine.

Uslovne ekstremne vrednosti

U zadacima 3291 — 3296 naći ekstremne vrednosti funkcija.

3291. $z=x^m+y^m\ (m>1)$ za $x+y=2$ $(x>0,\ y>0)$.

3292. $z=xy$ za $x^2+y^2=2a^2$.

3293. $z=\frac{1}{x}+\frac{1}{y}$ za $\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{a^2}$.

3294. $z=a\cos^2x+b\cos^2y$ za $y-x=\frac{\pi}{4}$.

3295. $u=x+y+z$ za $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$.

3296. $u=xyz$ za $\begin{cases} 1)\ x+y+z=5, \\ 2)\ xy+xz+yz=8. \end{cases}$

3297*. Dokazati da važi nejednakost

$$\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}>\left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^2.$$

3298. $f(x, y)=x^3-3xy^2+18y$, pri čemu je $3x^2y-y^3-6x=0$. Dokazati da funkcija $f(x, y)$ dostiže ekstremum u tačkama $x=y=\pm\sqrt{3}$.

3299. Naći minimum funkcije $u=ax^2+by^2+cz^2$, pri čemu su a, b, c pozitivne konstante, a x, y, z su vezani realizacijom $x+y+z=1$.

3300. Naći najveću i najmanju vrednost funkcije

$$u=y^2+4z^2-4yz-2xz-2xy$$

pod uslovom $2x^2+3y^2+6z^2=1$.

3301. U ravni $3x-2z=0$ naći tačku za koju je zbir kvadrata odstojanja od $A(1, 1, 1)$ i $B(2, 3, 4)$ minimalan.

3302. U ravni $x+y-2z=0$ naći tačku za koju je zbir kvadrata odstojanja od ravni $x+3z=6$ i $y+3z=2$ minimalan.

3303. Date su tačke $A(4, 0, 4)$, $B(4, 4, 4)$, $C(4, 4, 0)$. Na površini lopte $x^2+y^2+z^2=4$ naći tačku S tako da zapremina piramide $SABC$ bude: a) maksimalna, b) minimalna. Proveriti tačnost rezultata metodama elementarne geometrije.

3304. Naći pravougli paralelepiped date zapremine V čija je površina minimalna.

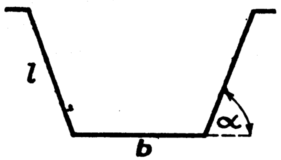
3305. Naći pravougli paralelepiped date površine S čija je zapremina maksimalna.

3306. Naći zapreminu najvećeg pravouglog paralelepipeda koji se može upisati u elipsoid sa poluosama a, b i c.

3307. Šator date zapremine ima oblik cilindra sa konusnim završetkom. U kom odnosu moraju stajati dimenzije šatora da bi količina materijala, potrebnog za njegovu izradu, bila minimalna?

3308. Presek kanala ima oblik jednakokrakog trapeza date površine; kolike moraju biti njegove dimenzije da bi kvašena površina kanala bila najmanja? (sl. 61)

3309. Od svih pravouglih paralelepipeda koji imaju datu dijagonalu naći onaj čija je zapremina maksimalna.



Sl. 61

3310. Odrediti spoljne dimenzije otvorenog (bez poklopca) sanduka koji ima oblik pravouglog paralelepipeda sa datom debljinom zidova α i datom zapreminom V, tako da bi količina materijala potrebnog za njegovu izradu bila minimalna.

3311. Odrediti paralelepiped najveće zapremine čiji zbir svih 12 ivica ima datu vrednost $(12a)$.

3312. Oko date elipse opisati trougao najmanje površine, čija je osnovica paralelna velikoj osi elipse.

3313. Na elipsi $\frac{x^2}{4}+\frac{y^2}{9}=1$ naći tačku čije je odstojanje od prave $3x-y-9=0$ minimalno, odnosno maksimalno.

3314. Na paraboli $x^2+2xy+y^2+4y=0$ naći tačku najbližu pravoj $3x-6y+4=0$.

3315. Na paraboli $2x^2-4xy+2y^2-x-y=0$ naći tačku najbližu pravoj $9x-7y+16=0$.

3316. Naći maksimalno odstojanje tačaka površine

$$2x^2+3y^2+2z^2=6$$

od ravni $z=0$.

3317. Naći stranice pravouglog trougla date površine S čiji je obim minimalan.

3318. U prav eliptični konus čije su poluose osnove a i b cm, a visina H cm, upisana je prizma sa pravougaonom osnovom tako da su osnovne ivice paralelne osama elipse, a presek dijagonala osnove leži u centru elipse. Kolike moraju biti osnovne ivice i visina prizme da bi njena zapremina bila maksimalna, i kolika je ta maksimalna zapremina?

3319. Naći pravilnu trostranu piramidu date zapremine, čiji je zbir svih ivica minimalan.

3320. Date su dve tačke elipse; odrediti položaj treće tačke elipse tako da površina trougla čija su temena pomenute tačke — bude maksimalna.

3321. Odrediti onu normalu elipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ čije je odstojanje od koordinatnog početka maksimalno.

3322. Na obrtnom elipsoidu $\frac{x^2}{96}+y^2+z^2=1$ naći tačku čije je odstojanje od ravni $3x+4y+12z=288$ minimalno, odnosno maksimalno.

3323. Date su ravne krive $f(x, y)=0$ i $\varphi(x, y)=0$. Pokazati da će rastojanje između tačaka (α, β) i (ξ, η), od kojih prva leži na prvoj a druga na drugoj krivoj, imati ekstremnu vrednost ako su ispunjeni sledeći uslovi.

$$\frac{\alpha-\xi}{\beta-\eta}=\frac{\left(\frac{\partial f}{\partial x}\right)_{\substack{x=\alpha\\y=\beta}}}{\left(\frac{\partial f}{\partial y}\right)_{\substack{x=\alpha\\y=\beta}}}=\frac{\left(\frac{\partial \varphi}{\partial x}\right)_{\substack{x=\xi\\y=\eta}}}{\left(\frac{\partial \varphi}{\partial y}\right)_{\substack{x=\xi\\y=\eta}}}.$$

Koristeći se ovim rezultatom naći najkraće rastojanje između elipse $x^2+2xy+5y^2-16y=0$ i prave $x+y-8=0$.

Rješenja

3259. $(0, 0), \left(-\frac{5}{3}, 0\right), (-1, 2), (-1, -2)$.

3260. $\left(\frac{1}{2}, -1\right)$. **3261.** $(0, 0), (0, a), (a, 0), \left(\frac{a}{3}, \frac{a}{3}\right)$.

3262. $(0, 0), (0, 2b), (a, b), (2a, 0), (2a, 2b)$. **3263.** $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$.

3264. $\left(\frac{b}{a}, \frac{c}{a}\right)$. **3265.** $\left(-\frac{2}{3}, -\frac{2}{3}\right)$. **3266.** $(2, 1, 7)$. **3267.** $(6, 4, 10)$.

3268. A i C su tačke maksimuma, B — tačka minimuma; u okolini tačke D površ ima oblik „sedla", duž prave EF funkcija zadržava konstantnu vrednost.

3269. $(-2, 0), \left(\frac{16}{7}, 0\right)$. **3270.** $(1, 1), (-1, -1)$.

3271*. Da bismo se uverili da je nađena tačka — tačka maksimuma dovoljno je predstaviti funkciju u obliku $z=10-(x-y)^2-2x^2-y^2$.

3272. $(2, -2)$. **3273.** $(-1, 1)$. **3277.** U tački $(6,4)$ funkcija dostiže maksimum.

3278. U tački $(0, 0)$ nema ekstremuma; u tački $(1, 1)$ funkcija dostiže minimum.

3279. Najveću i najmanju vrednost funkcija dostiže na granici oblasti: najveću $z=4$ u taćkama $(2, 0)$ i $(-2, 0$, a najmanju, $z=-4$, u tačkama $(0, 2)$ i $(0, -2)$. U stacionarnoj tački $(0, 0)$ nema ekstremuma.

3280. Najveća vrednost $z=17$ u tački $(1, 2)$; najmanja vrednost $z=-3$ u tački $(1, 0)$; stacionarna tačka $(-4, 6)$ leži van date oblasti.

3281. Najveća vrednost $z=4$ u stacionarnoj tački $(2, 1)$ (ova tačka je, prema tome, tačka ekstremuma); najmanja vrednost $z=-64$ u tački $(4, 2)$ koja leži na granici oblasti.

3282. Najmanju vrednost $z=0$ funkcija dostiže u tački $(0, 0)$; najveću vrednost $z=\frac{3}{e}$ u tačkama $(0, \pm 1)$.

3283. $z_{\max}=\frac{3}{2}\sqrt{3}$ u tački $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $z_{\min}=0$ u tački $(0, 0)$ (na granici oblasti).

3284. Svi sabirci moraju biti jednaki među sobom.

3285. Svi množitelji moraju biti jednaki među sobom.

3286. $\left(\frac{8}{5}, \frac{16}{5}\right)$. **3287.** $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3$.

3288. $x=\frac{\sum_{i=1}^{n} x_i}{n}$, $y=\frac{\sum_{i=1}^{n} y_i}{n}$. **3289.** $(3, \sqrt{39}, 0); (3, -\sqrt{39}, 0)$.

3290. Kocka. **3291.** U tački $(1, 1)$ je $z=2$ — minimum.

3292. (a, a) ili $(-a, -a)$, $z=a^2$ (maksimum), $(a, -a)$ ili $(-a, a)$, $z=-a^2$ (minimum).

3293. $(-a\sqrt{2}, -a\sqrt{2})$, $z=-\frac{\sqrt{2}}{a}$ (ninimum), $(a\sqrt{2}, a\sqrt{2})$, $z=\frac{\sqrt{2}}{a}$ (maksimum).

3294. Stacionarne tačke $x=-\frac{1}{2}\operatorname{Arctg}\frac{b}{a}$, $y=\frac{1}{2}\operatorname{Arctg}\frac{a}{b}$.

3295. $(3, 3, 3), u=9$ (minimum).

3296. Kad su vrednosti dveju nezavisno promenljivih $=2$, a vrednost treće $=1$, funkcija dostiže minimum $=4$; kad su vrednosti dveju nezavisno promenljivih $=\frac{4}{3}$, a vrednost treće $\frac{7}{3}$, funkcija dostiže maksimum $=\frac{112}{27}$.

3297*. Treba naći minimum funkcije $\frac{x_1^2+x_2^2+\cdots+x_n^2}{n}$ pod uslovom $x_1+x_2+\cdots+x_n=A$. Uopšte, važi relacija $\frac{\Sigma x_i^k}{n}>\left(\frac{\Sigma x_i}{n}\right)^k$. za $k>1$.

3299. $u_{\min}=\frac{abc}{bc+ca+ab}$ za $x=\frac{bc}{bc+ca+ab}$; $y=\frac{ac}{bc+ac+ab}$; $z=\frac{ab}{bc+ac+ab}$.

3300. $x=\pm\frac{1}{2}$, $y=\pm\frac{1}{3}$, $z=\pm\frac{1}{6}$. **3301.** $\left(\frac{21}{13}, 2, \frac{63}{26}\right)$.

3302. $(3, -1, 1)$. **3303.** a) $(-2, 0, 0)$; b) $(2, 0, 0)$.

3304. Kocka. **3305.** Kocka. **3306.** $\frac{8abc}{3\sqrt{3}}$.

3307. Ako je R poluprečnik osnove šatora, H — visina cilindričnog dela, a h — visina konusnog završetka, onda moraju važiti sledeće relacije: $R=\frac{h\sqrt{5}}{2}$, $H=\frac{h}{2}$.

3308. Ako je b osnovica, l — krak, a α — ugao na osnoviii trapeza, onda mora biti $l=b=\frac{2\sqrt{A}}{\sqrt[4]{3}}$, $\alpha=\frac{\pi}{3}$, pri čemu je A data površina preseka; tada je kvašena površina $u=2\sqrt[4]{3}\cdot\sqrt{A}\approx 2{,}632\sqrt{A}$.

3309. Kocka. **3310.** Svaka od osnovnih ivica je $2\alpha+\sqrt{2v}$, a visina je dva puta manja $\alpha+\frac{1}{2}\sqrt[3]{2v}$.

3311. a^3 (kocka). **3312.** Minimalna površina ima vrednost $3\sqrt{3}\,ab$.

3313. $x=\pm\frac{4}{\sqrt{5}}$, $y=\pm\frac{3}{\sqrt{5}}$. **3314.** $\left(-\frac{5}{9} - \frac{1}{9}\right)$. **3315.** $(3, 5)$. **3316.** $z_{\max}=2$.

3317. Stranice trougla su $\sqrt{2S}$, $\sqrt{2S}$ i $2\sqrt{S}$.

3318. Visina je $\frac{H}{3}$, osnovne ivice su $\frac{2a\sqrt{2}}{3}$ i $\frac{2b\sqrt{2}}{3}$, a zapremina $V=\frac{8}{27}abH$.

3319. Tetraedar.

3320. Normala elipse u traženoj tački mora biti normalna na pravoj koja spaja date tačke.

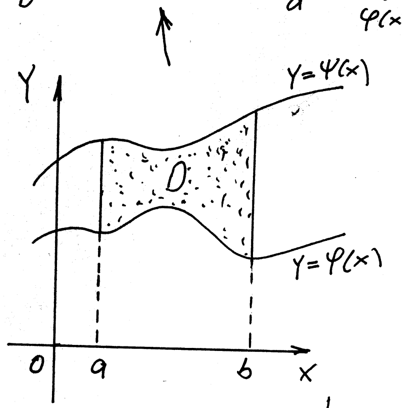
3321. Normalu povući u tački sa koordinatama

$$\left(\pm a\sqrt{\frac{a}{a+b}}, \pm b\sqrt{\frac{b}{a+b}}\right).$$

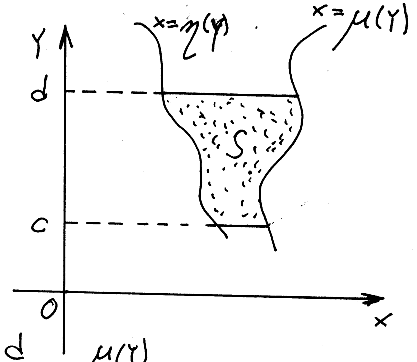
3322. $\left(9, \frac{1}{8}, \frac{3}{8}\right); \left(-9, -\frac{1}{8}, -\frac{3}{8}\right)$. **3323.** $2\sqrt{2}$.

Dvostruki integral

$$\iint\limits_{D} f(x,y)\,dx\,dy = \int_a^b dx \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy = \int_a^b \left[\int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy \right] dx$$



D-oblast integracije



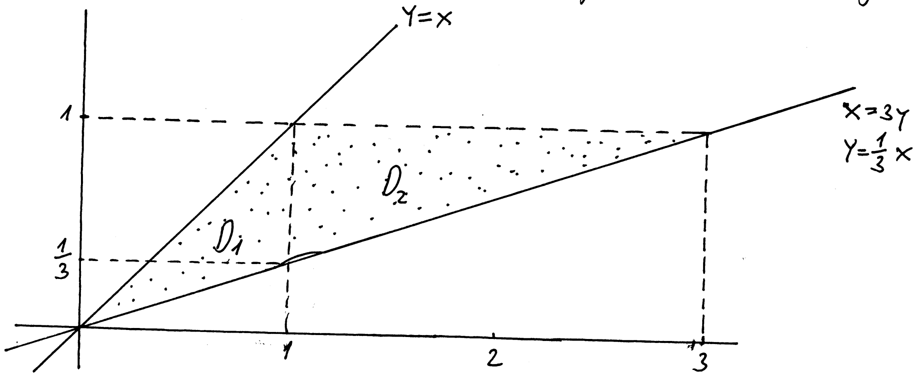
$$\iint\limits_{S} f(x,y)\,dx\,dy = \int_c^d dy \int_{\eta(y)}^{\mu(y)} f(x,y)\,dx = \int_c^d \left[\int_{\eta(y)}^{\mu(y)} f(x,y)\,dx \right] dy$$

(#) Izmeniti poredak integracije u integralu

$$I = \int_0^1 dy \int_y^{3y} f(x,y)\,dx \,.$$

Rj.

$x=3y$ i $x=y$ su prave. Skicirajmo oblast integracije



$$D: \begin{cases} 0 \le y \le 1 \\ y \le x \le 3y \end{cases} = D_1 \cup D_2 \qquad D_1: \begin{cases} 0 \le x \le 1 \\ \frac{1}{3}x \le y \le x \end{cases}$$

$$D_2: \begin{cases} 1 \le x \le 3 \\ \frac{1}{3}x \le y \le 1 \end{cases}$$

$$\int_0^1 dy \int_y^{3y} f(x,y)\,dx = \int_0^1 dx \int_{\frac{1}{3}x}^{x} f(x,y)\,dy + \int_1^3 dx \int_{\frac{1}{3}x}^{1} f(x,y)\,dy$$

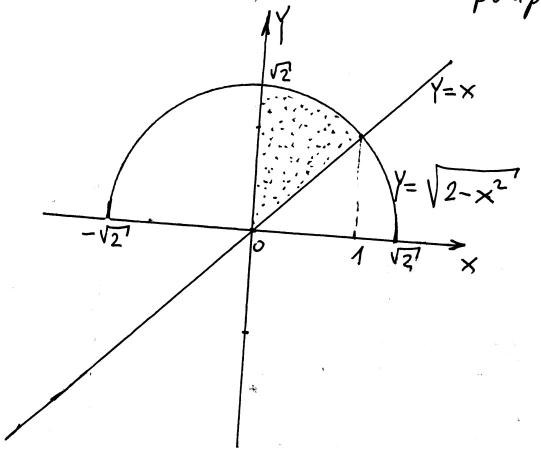
Izmeniti poredak integracije u integralu

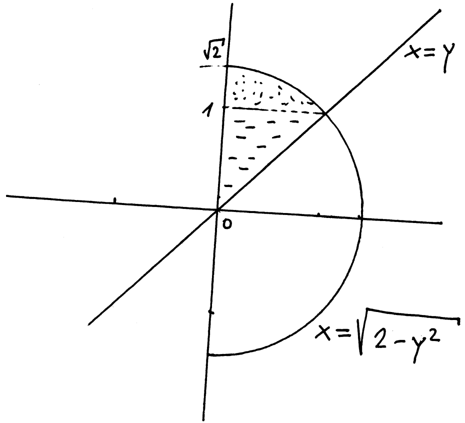
$$I=\int_0^1 dx \int_x^{\sqrt{2-x^2}} f(x,y)\, dy .$$

Rj. $y=x$ prava $\qquad y^2=2-x^2$

$y=\sqrt{2-x^2}$ parabola $\qquad x^2+y^2=2$

krug sa centrom u tački $(0,0)$
poluprečnika $r=\sqrt{2}\approx 1{,}41$



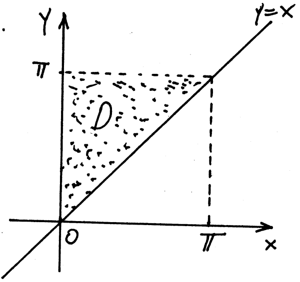


$$I=\int_0^1 dx \int_x^{\sqrt{2-x^2}} f(x,y)\, dy = \int_0^1 dy \int_0^y f(x,y)\, dx + \int_1^{\sqrt{2}} dy \int_0^{\sqrt{2-y^2}} f(x,y)\, dx$$

Izračunati dvostruki integral $\iint_D \cos(x+y)\, dx\, dy$

ako je $D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le \pi \ ; \ x\le y\le \pi\}$

Rj.



$$\iint_D \cos(x+y)\, dx\, dy = \int_0^\pi dx \int_x^\pi \cos(x+y)\, dy =$$

$$= \int_0^\pi dx \sin(x+y)\Big|_x^\pi = \int_0^\pi [\sin(x+\pi)-\sin 2x]\, dx \overset{(*)}{=}$$

$$\sin(x+\pi)=\sin x\cos\pi+\sin\pi\cos x=-\sin x$$

$$\overset{(*)}{=} \int_0^\pi(-\sin x-\sin 2x)\, dx = -\int_0^\pi \sin x\, dx - \int_0^\pi \sin 2x\, dx = \cos x\Big|_0^\pi + \frac{1}{2}\cos 2x\Big|_0^\pi =$$

$$= (-1-1)+\frac{1}{2}(1-1) = -2$$

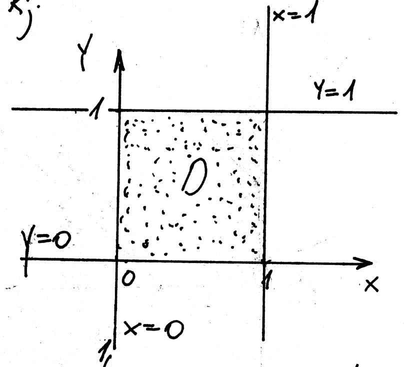
II način

$$\iint_D \cos(x+y)\, dx\, dy = \int_0^\pi dy \int_0^y \cos(x+y)\, dx = \int_0^\pi dy \sin(x+y)\Big|_0^y =$$

$$\int_0^\pi (\sin 2y-\sin y)\, dy = -\frac{1}{2}\cos 2y\Big|_0^\pi + \cos y\Big|_0^\pi = -\frac{1}{2}(1-1)+(-1-1)=-2$$

(#) Izračunati vrijednost integrala $I=\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy$ gdje je $D=\{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 1;\ 0\le y\le 1\}$.

Rj.:



I način

$$\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy = \int_0^1 dx\int_0^1 \frac{x^2}{1+y^2}\,dy = \int_0^1 x^2\,dx\int_0^1 \frac{dy}{1+y^2} = \int_0^1 x^2 \operatorname{arctg} y\Big|_0^1 dx =$$

$$= \frac{\pi}{4}\int_0^1 x^2\,dx = \frac{\pi}{4}\cdot\frac{x^3}{3}\Big|_0^1 = \frac{\pi}{12}$$

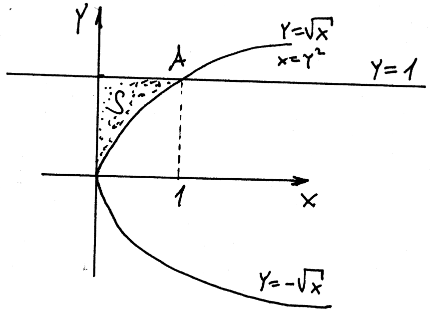
II način

$$\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy = \int_0^1 dy\int_0^1 \frac{x^2}{1+y^2}\,dx = \int_0^1 \frac{dy}{1+y^2}\int_0^1 x^2\,dx$$

$$= \int_0^1 \frac{1}{1+y^2}\cdot\frac{x^3}{3}\Big|_0^1 dy = \frac{1}{3}\int_0^1 \frac{dy}{1+y^2} = \frac{1}{3}\operatorname{arctg} x\Big|_0^1 = \frac{1}{3}\cdot\frac{\pi}{4} = \frac{\pi}{12}$$

(#) Izračunati integral $\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy$, gdje je S oblast omeđena parabolom $y^2=x$, te pravama $x=0$, $y=1$.

Rj. Nacrtajmo sliku



Tačka $A(1,1)$ je presjek parabole $y^2=x$ i prave $y=1$.

Moguća su dva načina:

$$\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy = \int_0^1\Big[\int_0^{y^2} e^{-\frac{x}{y}}\,dx\Big]dy$$

ili

$$\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy = \int_0^1\Big[\int_{\sqrt{x}}^{1} e^{-\frac{x}{y}}\,dy\Big]dx$$

Kako $\int e^{-\frac{1}{t}}\,dt = ?$ imamo:

$$\iint\limits_S e^{-\frac{x}{y}}\,dx\,dy = \int_0^1\Big[\int_0^{y^2} e^{-\frac{x}{y}}\,dx\Big]dy = \int_0^1 \Big(-y\, e^{-\frac{x}{y}}\Big|_0^{y^2}\Big)dy = -\int_0^1 y(e^{-y}-1)\,dy \overset{(*)}{=}$$

$$\int e^{-\frac{x}{y}}\,dx = \left|\begin{matrix} -\frac{x}{y}=t \\ -\frac{1}{y}dx=dt \end{matrix}\right| = \int e^{t}(-y)\,dt = -y\,e^{t}+c = -y\,e^{-\frac{x}{y}}+c$$

$$\overset{(*)}{=} \int_0^1 (y-ye^{-y})\,dy = \int_0^1 y\,dy + \int_0^1 (-y)e^{-y}\,dy = \frac{1}{2}y^2\Big|_0^1 + (y+1)\,e^{-y}\Big|_0^1 \overset{(\Delta)}{=}$$

$$\int t e^{t}\,dt = \left|\begin{matrix} u=t & dv=e^{t}dt \\ du=dt & v=e^{t} \end{matrix}\right| = te^{t}-\int e^{t}dt = (t-1)e^{t}+c$$

$$\int (-t)e^{-t}\,dt = \left|\begin{matrix} u=-t & dv=e^{-t}dt \\ du=-dt & v=-e^{-t} \end{matrix}\right| = te^{-t}-\int e^{-t}dt = (t+1)e^{-t}+c$$

$$\overset{(\Delta)}{=} \frac{1}{2} + (2e^{-1}-1) = \frac{2}{e}-\frac{1}{2}$$

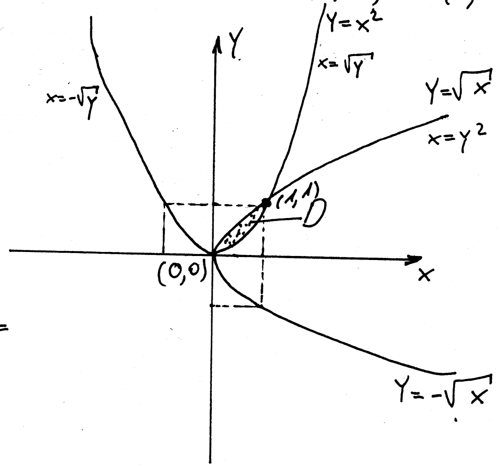
(#) Izračunati dvostruki integral $\iint\limits_D (x^2+y)\,dx\,dy$

gdje je D površ ograničena linijama $y=x^2$ i $y^2=x$.

Rj. Nađimo presječnu tačku i nacrtajmo ove dvije krive

$$\begin{array}{c} y=x^2 \\ y^2=x \\ \hline x^4=x \\ x(x^3-1)=0 \\ x(x-1)(x^2+x+1)=0 \\ x=0 \text{ ili } x=1 \end{array}$$

Presječne tačke krivih su (0,0) i (1,1).



$$\iint\limits_D (x^2+y)\,dx\,dy=\int_0^1 dx\int_{x^2}^{\sqrt{x}}(x^2+y)\,dy=$$

$$=\int_0^1 dx\left(x^2y\Big|_{x^2}^{\sqrt{x}}+\frac12 y^2\Big|_{x^2}^{\sqrt{x}}\right)=\int_0^1\left[x^2(\sqrt{x}-x^2)+\frac12(x-x^4)\right]dx$$

$$=\int_0^1\left(x^2\sqrt{x}\underline{-x^4}+\frac12x-\underline{\frac12x^4}\right)dx=\int_0^1\left(-\frac32x^4+x^{\frac52}+\frac12x\right)dx=$$

$$=-\frac32\cdot\frac15x^5\Big|_0^1+\frac27\cdot x^{\frac72}\Big|_0^1+\frac12\cdot\frac12x^2\Big|_0^1=-\frac{3}{10}+\frac27+\frac14=\frac{-3\cdot14+2\cdot20+1\cdot35}{140}$$

$$=\frac{-42+40+35}{140}=\frac{33}{140}$$

II način: $\iint\limits_D(x^2+y)\,dx\,dy=\int_0^1\left(\int_{y^2}^{\sqrt{y}}(x^2+y)\,dx\right)dy=\ldots=\int_0^1\left(\frac43\sqrt{y^3}-y^3-\frac13y^6\right)dy$

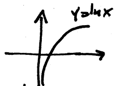
↗ za vježbu

$$=\ldots=\frac{33}{140}$$

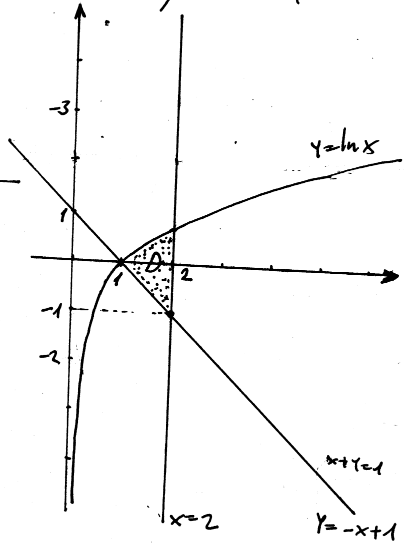
(#) Izračunati dvostruki integral $I=\iint\limits_D xy\,dx\,dy$,

gdje je $D:\ y=\ln x,\ \ x=2,\ \ x+y=1.$

Rj. Kriva $y=\ln x$ izgleda ovako



Nacrtajmo sliku



Pronađimo presječne tačke datih krivi.

$$\begin{array}{c} y=\ln x \\ x=2 \\ \hline y=\ln2\approx0{,}69 \\ (2,\ln2) \end{array}\qquad \begin{array}{c} y=\ln x \\ x+y=1 \\ \hline y=\ln x \\ y=-x+1 \\ \hline \ln x=-x+1 \\ x=1 \\ (1,0) \end{array}\qquad \begin{array}{c} x=2 \\ x+y=1 \\ \hline 2+y=1 \\ y=-1 \\ (2,-1) \end{array}$$

$$I=\iint\limits_D xy\,dx\,dy=\int_1^2dx\int_{-x+1}^{\ln x}xy\,dy=\int_1^2x\,dx\int_{-x+1}^{\ln x}y\,dy=$$

$$=\int_1^2x\left(\frac12y^2\Big|_{-x+1}^{\ln x}\right)dx=\frac12\int_1^2x\Big(\ln^2x-\underbrace{(-x+1)^2}_{x^2-2x+1}\Big)dx=$$

$$=\frac12\int_1^2x\ln^2x\,dx-\frac12\int_1^2(x^3-2x^2+x)\,dx$$

$$\int_1^2x\ln^2x\,dx=\begin{vmatrix}u=\ln^2x & dv=x\,dx\\ du=2\ln x\cdot\frac1x\,dx & v=\frac12x^2\end{vmatrix}=\overset{2\cdot0{,}48\approx0{,}96}{\overbrace{\frac12x^2\ln^2x\Big|_1^2}^{\frac12 4(\ln2)^2}}-\int_1^2x\ln x\,dx=$$

$$=\begin{vmatrix}u=\ln x & dv=x\,dx\\ du=\frac1x\,dx & v=\frac12x^2\end{vmatrix}=2\ln^22-\Big[\overbrace{\frac12x^2\ln x\Big|_1^2}^{2\ln2}-\frac12\underset{\substack{\frac12x^2|_1^2\\ \frac12\cdot3}}{\int_1^2x\,dx}\Big]=2\ln^22-2\ln2+\frac34$$

$$\int_1^2(x^3-2x^2+x)\,dx=\frac14x^4\Big|_1^2-\frac23x^3\Big|_1^2+\frac12x^2\Big|_1^2=$$

$$=\frac{15}{4}-\frac{14}{3}+\frac32=\frac{45-56+18}{12}=\frac{7}{12}$$

traženo rješenje ↙

$$I=\frac12\left(2\ln^22-2\ln2+\frac34\right)-\frac12\cdot\frac{7}{12}=\ln^22-\ln2+\frac38-\frac{7}{24}=\ln^22-\ln2+\frac{1}{12}$$

㈱ Izračunati dvostruki integral $I=\iint\limits_D xy\,dx\,dy$, gdje je D oblast ograničena linijama $xy=1$, $x+y=\frac{5}{2}$.

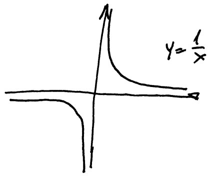
Rj. Skicirajmo oblast D

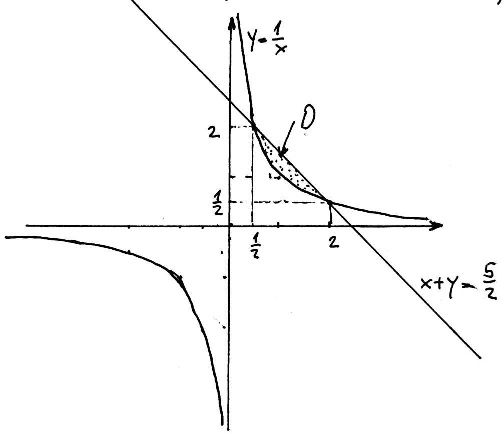
$xy=1$

$y=\frac{1}{x}$ $\mathcal{D}: x\in\mathbb{R}\setminus\{0\}$
f-ja je neparna (simetrična u odnosu na 0)
ne siječe y-osu, ne siječe x-osu

$\lim\limits_{x\to 0+}\frac{1}{x}=+\infty$

$\lim\limits_{x\to\infty}\frac{1}{x}=0$





Naćimo presječne tačke krive $xy=1$ i prave $x+y=\frac{5}{2}$.

$$\frac{\begin{array}{l} xy=1 \\ x+y=\frac{5}{2}\end{array}}{\begin{array}{l} y=\frac{1}{x} \\ x+y=\frac{5}{2}\end{array}}$$

$x+\frac{1}{x}=\frac{5}{2}$ $/\cdot x$

$x^2-\frac{5}{2}x+1=0$ $/\cdot 2$

$2x^2-5x+2=0$

$D=25-16=9$

$x_{1,2}=\frac{5\pm 3}{4}$ $x_1=\frac{2}{4}=\frac{1}{2}$

$x_2=2$

$x_1=\frac{1}{2} \Rightarrow y_1=2$

$x_2=2 \Rightarrow y=\frac{1}{2}$

$$D:\begin{cases}\frac{1}{2}<x<2\\ \frac{1}{x}<y<\frac{5}{2}-x\end{cases}$$

$$\iint\limits_D xy\,dx\,dy=\int_{\frac{1}{2}}^{2}x\,dx\int_{\frac{1}{x}}^{\frac{5}{2}-x}y\,dy=$$

$$=\int_{\frac{1}{2}}^{2}x\,\frac{1}{2}y^2\Big|_{\frac{1}{x}}^{\frac{5}{2}-x}dx=\frac{1}{2}\int_{\frac{1}{2}}^{2}x\left(\left(\frac{5}{2}-x\right)^2-\frac{1}{x^2}\right)dx=\frac{1}{2}\int_{1/2}^{2}x\left(\frac{25}{4}-5x+x^2-\frac{1}{x^2}\right)dx$$

$$\frac{1}{2}\int_{1/2}^{2}\left(\frac{25}{4}x-5x^2+x^3-\frac{1}{x}\right)dx=\int_{\frac{1}{2}}^{2}\left(\frac{1}{2}x^3-\frac{5}{2}x^2+\frac{25}{8}x-\frac{1}{2}\cdot\frac{1}{x}\right)dx=\ldots=\frac{165}{128}-\ln 2$$

rješenje

㈱ Promjeniti poredak integracije u integralu

$$I=\int_{-7}^{1}dy\int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}}f(x,y)\,dx$$

Rj. Ako sa D označimo oblast integracije imamo

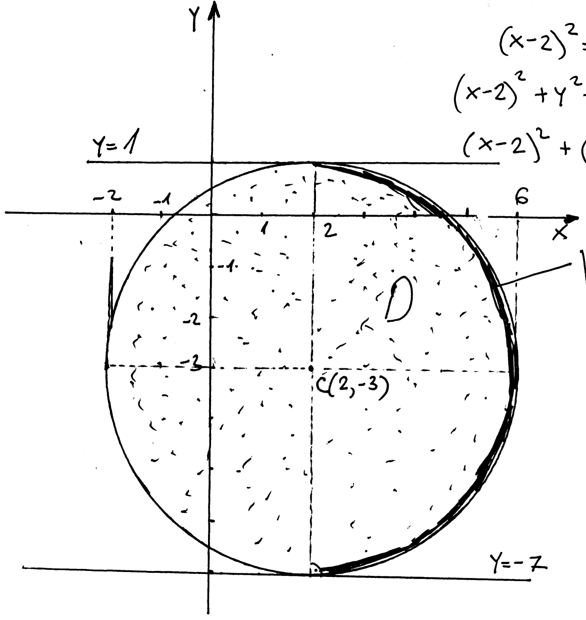
$$D:\begin{cases}2-\sqrt{7-6y-y^2}\le x\le 2+\sqrt{7-6y-y^2}\\ -7\le y\le 1\end{cases}$$

Posmatrajmo prvo samo promjenljivu x: $-\sqrt{7-6y-y^2}\le x-2\le\sqrt{7-6y-y^2}$

$(x-2)^2=7-6y-y^2$

$(x-2)^2+y^2+2\cdot 3\cdot y+9-9=7$

$(x-2)^2+(y+3)^2=16$ krug sa centrom u tački C(2,-3) poluprečnika r=4



$$(y+3)^2=16-\underbrace{(x-2)^2}_{x^2-4x+4}$$

$(y+3)^2=12-x^2+4x$

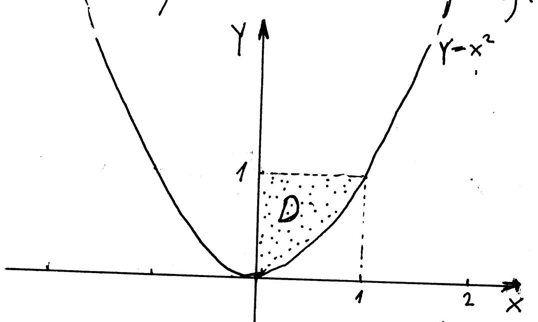
$y_{1,2}=-3\pm\sqrt{12-x^2+4x}$

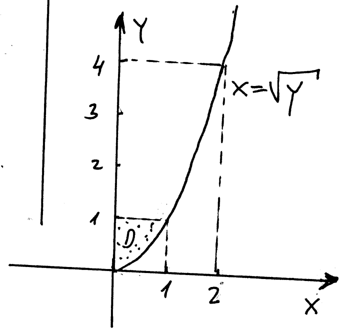
Prema tome

$$I=\int_{-7}^{1}dy\int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}}f(x,y)\,dx=\int_{-2}^{6}dx\int_{-3-\sqrt{12-x^2+4x}}^{-3+\sqrt{12-x^2+4x}}f(x,y)\,dx.$$

Izračunati integral $I=\int_0^1 x^5 dx \int_{x^2}^1 e^{y^2} dy$.

Rj. Skicirajmo oblast integracije D





$$I=\int_0^1 x^5 dx \int_{x^2}^1 e^{y^2} dy =$$

$$=\iint_D x^5 e^{y^2} dx\, dy =$$

$$=\int_0^1 e^{y^2} dy \int_0^{\sqrt{y}} x^5 dx =$$

$$=\int_0^1 e^{y^2} \frac{1}{6} x^6 \Big|_0^{\sqrt{y}} dy =$$

$$=\frac{1}{6}\int_0^1 e^{y^2} y^3 dy = \left| \begin{matrix} u=y^2 & dv=e^{y^2} y\, dy = \frac{1}{2} e^{y^2} d(y^2) \\ du=2y\, dy & v=\frac{1}{2} e^{y^2} \end{matrix} \right| =$$

$$=\frac{1}{12} y^2 e^{y^2}\Big|_0^1 - \frac{1}{6}\int_0^1 e^{y^2} y\, dy = \frac{1}{12}\left(1\cdot e^1 - 0\right) - \frac{1}{12}\int_0^1 e^{y^2} d(y^2) =$$

$$=\frac{1}{12} e - \frac{1}{12} e^{y^2}\Big|_0^1 = \frac{1}{12} e - \frac{1}{12}(e-1) = \frac{1}{12} e - \frac{1}{12} e + \frac{1}{12} = \frac{1}{12}$$

traženo rješenje

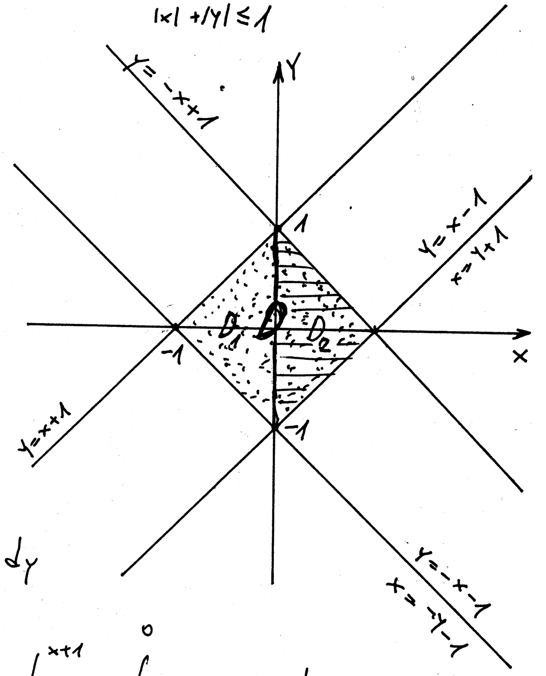
Izračunati vrijednost integrala $I=\iint_{|x|+|y|\leq 1} x^2 dx\, dy$

Rj.: $x<0,\ y<0 \Rightarrow -x-y\leq 1$
$y\geq -x-1$

$x<0,\ y>0 \Rightarrow -x+y\leq 1$
$y\leq x+1$

$x\geq 0,\ y<0 \Rightarrow x-y\leq 1$
$y\geq x-1$

$x\geq 0,\ y\geq 0 \Rightarrow x+y\leq 1$
$y\leq -x+1$



$$I=\iint_{|x|+|y|\leq 1} x^2 dx\, dy = \iint_{D_1} x^2 dx\, dy + \iint_{D_2} x^2 dx\, dy$$

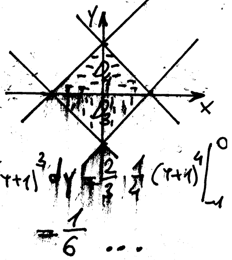
$$\iint_{D_1} x^2 dx\, dy = \int_{-1}^0 dx \int_{-x-1}^{x+1} x^2 dy = \int_{-1}^{} x^2 dx\, y\Big|_{-x-1}^{x+1} = \int_{-1}^0 x^2 (2x+2)\, dx =$$
$$\underbrace{}_{x+1+x+1}$$

$$=\int_{-1}^0 (2x^3+2x^2)\, dx = \frac{2}{4}x^4\Big|_{-1}^0 + \frac{2}{3}x^3\Big|_{-1}^0 = \frac{1}{2}\cdot(-1)+\frac{2}{3}\cdot 1 = \frac{2}{3}-\frac{1}{2} = \frac{1}{6}$$

$$\iint_{D_2} x^2 dx\, dy = \int_0^1 dx \int_{x-1}^{-x+1} x^2 dy = \int_0^1 x^2 dx\, y\Big|_{x-1}^{-x+1} = \int_0^1 x^2(-2x+2)\, dx =$$
$$-x+1-x+1$$

$$\int_0^1 (2x^3+2x^2)\, dx = -\frac{2}{4}x^4\Big|_0^1 + \frac{2}{3}x^2\Big|_0^1 = -\frac{1}{2}+\frac{2}{3} = \frac{1}{6},$$

$$I=\iint_{|x|+|y|\leq 1} x^2 dx\, dy = \frac{2}{6}$$

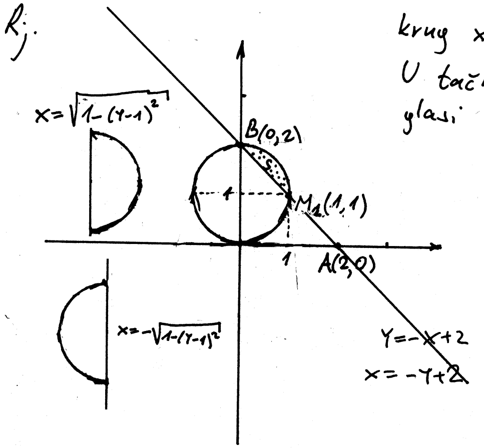


II način: $I=\iint_{D_3} x^2 dx\, dy + \iint_{D_4} x^2 dx\, dy$

$$\iint_{D_3} x^2 dx\, dy = \int_{-1}^0 dy \int_{-y-1}^{y+1} x^2 dx = \int_{-1}^0 dy \frac{1}{3} x^3\Big|_{-y-1}^{y+1} = \frac{1}{3}\int_{-1}^0 2(y+1)^3 dy = \frac{2}{3}\int_{-1}^0 (y+1)^3 dy = \frac{2}{3}\cdot\frac{1}{4}(y+1)^4\Big|_{-1}^0$$
$$(-1)(y+1)$$
$$=\frac{1}{6}\ \dots$$

(#) Izračunati dvostruki integral $\iint\limits_S x\,dx\,dy$ gdje je područje integracije S ograničeno pravcem koji prolazi tačkama A(2,0), B(0,2) i lukom kruga poluprečnika 1 sa centrom u tački C(0,1).

Rj.



$\overset{x_1,y_1}{A(2,0)}$, $\overset{x_2,y_2}{B(0,2)}$

krug $x^2+(y-1)^2=1$

U tačkama A(2,0) i B(0,2) jednačina prave glasi:

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y-0=\frac{2-0}{0-2}(x-2)$$

$$y=-x+2$$

Nađimo presječne tačke prave i kruga

$$x^2+(y-1)^2=1$$
$$y=-x+2$$
$$x^2+(-x+2-1)^2=1$$

$x^2+(-x+1)^2=1$

$x^2+x^2-2x+1=1$

$2x^2-2x=0$

$2x(x-1)=0$

$x_1=0$

$x_2=1$

$x_1=0 \Rightarrow y=2$

$x_2=1 \Rightarrow y=1$

Presječne tačke prave i kruga su $M_1(0,2)$ i $M_2(1,1)$

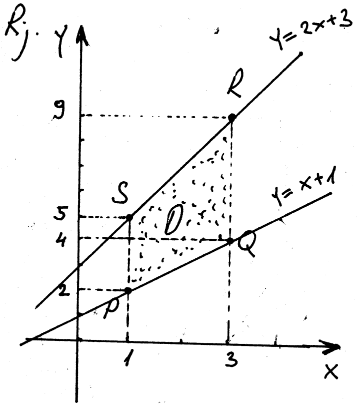
$$\iint\limits_S x\,dx\,dy=\int_1^2\left[\int_{-y+2}^{\sqrt{1-(y-1)^2}} x\,dx\right]dy=\int_1^2 \frac{1}{2}x^2\Big|_{-y+2}^{\sqrt{1-(y-1)^2}}dy=\frac{1}{2}\int_1^2\left[(1-(y-1)^2)-(2-y)^2\right]dy$$

$$=\frac{1}{2}\int_1^2\left[1-y^2+2y-1-4+4y-y^2\right]dy=\frac{1}{2}\int_1^2(-2y^2+6y-4)\,dy=\frac{1}{2}\cdot 2\int_1^2(-y^2+3y-2)\,dy$$

$$=-\frac{1}{3}y^3\Big|_1^2+\frac{3}{2}y^2\Big|_1^2-2y\Big|_1^2=-\frac{7}{3}+\frac{9}{2}-2=\frac{-14+27-12}{6}=\frac{1}{6}$$

(#) Izračunati $\iint\limits_D x\,dx\,dy$ pri čemu je D četverougao □PQRS gdje su tačke P(1,2), Q(3,4), R(3,9) i S(1,5).

Rj.



$$\iint\limits_D x\,dx\,dy=\int_1^3\left[\int_{x+1}^{2x+3} x\,dy\right]dx=$$

$y-y_1=k(x-x_1)$

$$y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$\overset{x_1\ y_1}{P(1,2)}$

$Q(3,4)$

$$y-2=\frac{2}{2}(x-1)$$

$$y=x+1$$

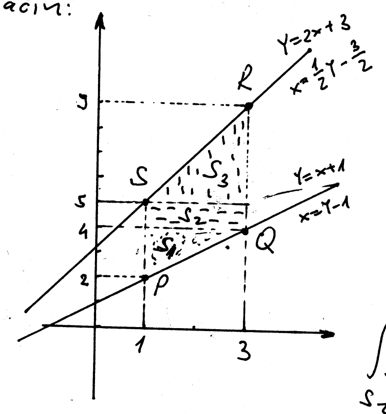
$\overset{x_1\ y_1}{R(3,9)}$

$S(1,5)$ $\quad y-9=\frac{-4}{-2}(x-3) \Rightarrow y=2x+3$

$$=\int_1^3 xy\Big|_{x+1}^{2x+3}dx=\int_1^3 x(2x+3-x-1)\,dx=\int_1^3(x^2+2x)\,dx=\frac{1}{3}x^3\Big|_1^3+x^2\Big|_1^3=$$

$$=\frac{1}{3}(27-1)+(9-1)=\frac{26}{3}+8=\frac{50}{3}$$

II način: $\quad$ S: □PQRS



$$\iint\limits_S x\,dx\,dy=\iint\limits_{S_1} x\,dx\,dy+\iint\limits_{S_2} x\,dx\,dy+\iint\limits_{S_3} x\,dx\,dy$$

$$\iint\limits_{S_1} x\,dx\,dy=\int_2^4\left[\int_1^{y-1} x\,dx\right]dy=\ldots=\int_2^4\left(\frac{1}{2}y^2-y\right)dy=\ldots=\frac{10}{3}$$

$$\iint\limits_{S_2} x\,dx\,dy=\int_4^5\left[\int_1^3 x\,dx\right]dy=\ldots=\int_4^5 4\,dy=\ldots=4$$

$$\iint\limits_{S_3} x\,dx\,dy=\int_5^9\left[\int_{\frac{1}{2}y-\frac{3}{2}}^{3} x\,dx\right]dy=\ldots=\int_5^9\left(\frac{9}{2}-\frac{(y-3)^2}{8}\right)dy=\ldots=\frac{28}{3}$$

(#) Izračunati $\iint\limits_D (x+y)\,dx\,dy$ ako je D oblast ograničena linijama $y^2=2x$, $x+y=4$ i $x+y=12$.

Rj. Nađimo presječne tačke ovih linija.

$$\begin{array}{l} y^2=2x \\ x+y=4 \\ \hline y^2=2x \\ y=4-x \\ \hline \end{array}$$

$$(4-x)^2=2x$$
$$16-8x+x^2=2x$$
$$x^2-10x+16=0$$
$$D=100-64=36$$
$$x_{1,2}=\frac{10\pm 6}{2} \qquad x_1=2 \quad x_2=8$$

$A(2,2)$
$B(8,-4)$

$$\begin{array}{l} y^2=2x \\ x+y=12 \\ \hline y^2=2x \\ y=12-x \\ \hline \end{array}$$

$$(12-x)^2=2x$$
$$144-24x+x^2=2x$$
$$x^2-26x+144=0$$

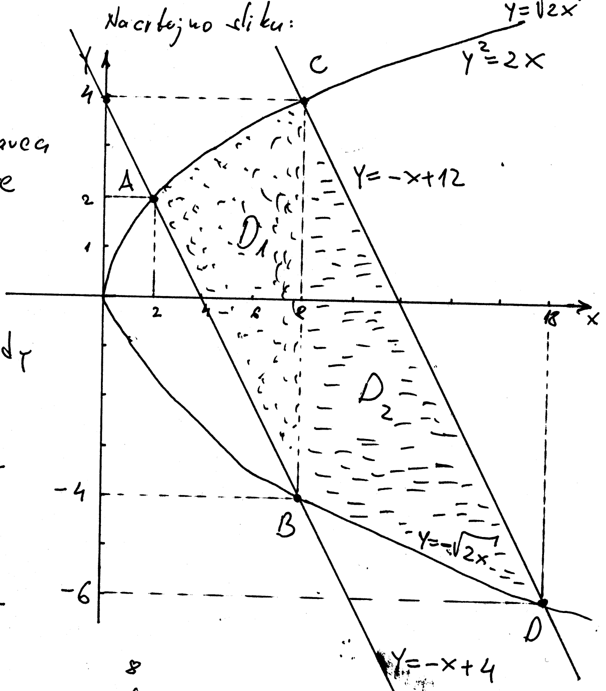
$$D=676-576$$
$$D=100$$
$$x_{1,2}=\frac{26\pm 10}{2}$$
$$x_1=8 \quad x_2=18$$

$C(8,4)$
$D(18,-6)$

$$\begin{array}{l} x+y=4 \\ x+y=12 \\ \hline y=-x+4 \\ y=-x+12 \end{array}$$

ove dvije prave imaju isti koeficijent pravca
dvije paralelne prave

Nacrtajmo sliku:



$$I=\iint\limits_D (x+y)\,dx\,dy=\iint\limits_{D_1}(x+y)\,dx\,dy+\iint\limits_{D_2}(x+y)\,dx\,dy$$

$$\iint\limits_{D_1}(x+y)\,dx\,dy=\int_2^8\left[\int_{-x+4}^{\sqrt{2}\cdot\sqrt{x}}(x+y)\,dy\right]dx=$$

$$=\int_2^8\left(xy\Big|_{-x+4}^{\sqrt{2}\cdot\sqrt{x}}+\frac{1}{2}y^2\Big|_{-x+4}^{\sqrt{2}\cdot\sqrt{x}}\right)dx=$$

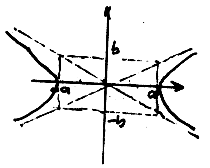
$$=\int_2^8\left[x\left(\sqrt{2}\sqrt{x}+x-4\right)+\frac{1}{2}\left(2x-(-x+4)^2\right)\right]dx=\dots=\int_2^8\left(x+\sqrt{2}\,x^{\frac{3}{2}}+\frac{1}{2}x^2-8\right)dx=\dots=\frac{826}{5}$$

$$\iint\limits_{D_2}(x+y)\,dx\,dy=\int_8^{18}\left[\int_{-\sqrt{2x}}^{-x+12}(x+y)\,dy\right]dx=\dots=\int_8^{18}\left(\sqrt{2}\,x^{\frac{3}{2}}-x-\frac{1}{2}x+72\right)dx=\dots=\frac{5678}{15}$$

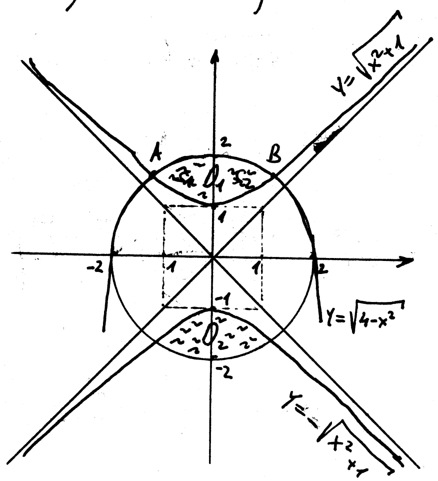
$I=543\frac{11}{15}$ vrijedn. dvostr. integr.

(#) Izračunati $I=\iint\limits_D dx\,dy$, ako je D: $y^2-x^2=1$, $x^2+y^2=4$.

Rj. Krive oblika $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ zovemo hiperbole i one su oblika



Skicirajmo naše dvije krive



$x^2+y^2=4$
je krug sa centrom u $(0,0)$
poluprečnikom $r=2$

$$D=D_1\cup D_2$$

$$I=\iint\limits_D dx\,dy=\iint\limits_{D_1\cup D_2}dx\,dy=2\iint\limits_{D_1}dx\,dy$$

$$y^2=4-x^2 \qquad y^2=x^2+1$$
$$y=\pm\sqrt{4-x^2} \qquad y=\pm\sqrt{x^2+1}$$

Nađimo presječnu tačku krivih $y=\sqrt{x^2+1}$ i $y=\sqrt{4-x^2}$

$$\sqrt{x^2+1}=\sqrt{4-x^2} \quad /^2$$
$$x^2+1=4-x^2 \qquad x_1=-\sqrt{\frac{3}{2}}$$
$$2x^2-3=0$$
$$2x^2=3 \qquad x_2=\sqrt{\frac{3}{2}}$$
$$x^2=\frac{3}{2}$$

$$x_1=-\sqrt{\frac{3}{2}} \Rightarrow y=\sqrt{\frac{3}{2}+1}=\sqrt{\frac{5}{2}}$$

Presječne tačke su $A\left(-\sqrt{\frac{3}{2}},\sqrt{\frac{5}{2}}\right)$ i $B\left(\sqrt{\frac{3}{2}},\sqrt{\frac{5}{2}}\right)$.

Primjetimo da je oblast D_1 simetrična

$$\iint\limits_{D_1}dx\,dy=\iint\limits_{S_1}dx\,dy+\iint\limits_{S_2}dx\,dy=2\iint\limits_{S_2}dx\,dy=2\int_0^{\sqrt{\frac{3}{2}}}dx\int_{\sqrt{x^2+1}}^{\sqrt{4-x^2}}dy=2\int_0^{\sqrt{\frac{3}{2}}}\left(\sqrt{4-x^2}-\sqrt{x^2+1}\right)dx$$

$$\int\sqrt{4-x^2}\,dx=2\arcsin\frac{x}{2}+\frac{x\sqrt{4-x^2}}{2} \quad \text{(sami)}$$

$$\int\sqrt{x^2+1}\,dx=\int\frac{x^2+1}{\sqrt{x^2+1}}\,dx\overset{\text{(metodom Ostrogradskog)}}{=}(ax+b)\sqrt{x^2+1}+\lambda\int\frac{dx}{\sqrt{x^2+1}} \quad /'$$

$$\frac{x^2+1}{\sqrt{x^2+1}} = a\sqrt{x^2+1} + (ax+b)\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}} + \lambda\frac{1}{\sqrt{x^2+1}} \quad /\cdot\sqrt{x^2+1}$$

$$x^2+1 = \underline{a}(x^2+1) + \underline{a}x^2+\underline{\underline{bx}} + \lambda$$

$2a=1 \quad \Rightarrow \quad a=\frac{1}{2}$

$b=0$

$a+\lambda=1 \quad\quad \lambda=\frac{1}{2}$

$$\int\sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\int\frac{dx}{\sqrt{x^2+1}}$$

$$\int\sqrt{x^2+1}\,dx = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln|x+\sqrt{x^2+1}|+C$$

$$\int_0^{\sqrt{\frac{3}{2}}}\sqrt{4-x^2}\,dx = 2\arcsin\frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{4} \quad \text{(sami)}$$

$$\int_0^{\sqrt{\frac{3}{2}}}\sqrt{x^2+1}\,dx = \frac{1}{2}\cdot\sqrt{\frac{3}{2}}\,\overbrace{\sqrt{\frac{3}{2}+1}}^{\sqrt{\frac{5}{2}}} + \frac{1}{2}\ln\left|\sqrt{\frac{3}{2}}+\sqrt{\frac{3}{2}+1}\right| = \frac{\sqrt{15}}{4} + \frac{1}{2}\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|$$

$$I = \iint_D dx\,dy = 2\iint_{D_1} dx\,dy = 2\cdot 2\iint_{S_2} dx\,dy = 4\left(2\arcsin\frac{\sqrt{6}}{4} + \frac{\sqrt{15}}{4} - \right.$$

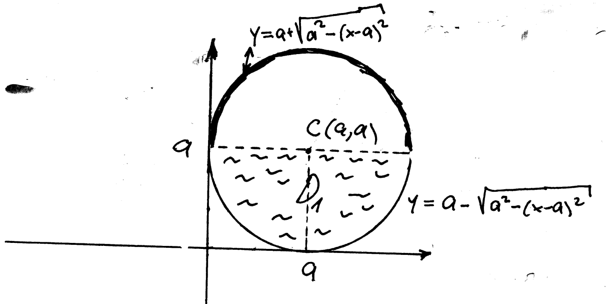
$$\left. \frac{\sqrt{15}}{4} - \frac{1}{2}\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|\right) = 8\arcsin\frac{\sqrt{6}}{4} - 2\ln\left|\frac{\sqrt{6}+\sqrt{10}}{2}\right|$$

traženo rješenje.

U rješenju sljedećeg zadatka ima greška, Pronađi grešku.

(#) Izračunati dvostruki integral $\iint_S \frac{dx\,dy}{\sqrt{2a-x}}$ gdje je S krug poluprečnika a, koji dodiruje koordinatne ose i leži u prvom kvadrantu.

Rj.



krug $(x-a)^2+(y-a)^2=a^2$

$$y-a=\pm\sqrt{a^2-(x-a)^2}$$

$$y=a\pm\sqrt{a^2-(x-a)^2}$$

$$\iint_S\frac{dx\,dy}{\sqrt{2a-x}} = 2D_1$$

$$D_1 = \int_0^{2a}\left[\int_{a-\sqrt{a^2-(x-a)^2}}^{a}\frac{1}{\sqrt{2a-x}}\,dy\right]dx =$$

$$= \int_0^{2a}\frac{dx}{\sqrt{2a-x}}\cdot y\Big|_{a-\sqrt{a^2-(x-a)^2}}^{a} = \int_0^{2a}\sqrt{\frac{a^2-(x^2-2ax+a^2)}{2a-x}}\,dx = \int_0^{2a}\sqrt{\frac{a^2-x^2+2ax-a^2}{2a-x}}\,dx$$

$$= \int_0^{2a}\sqrt{\frac{x(2a-x)}{2a-x}}\,dx = \int_0^{2a}\sqrt{x}\,dx = \int_0^{2a}x^{\frac{1}{2}}dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}}\Big|_0^{2a} = \frac{2}{3}\cdot(2a)^{\frac{3}{2}} =$$

$$= \frac{2}{3}\sqrt{8a^3} = \frac{2}{3}\,2a\sqrt{2a} = \frac{4a}{3}\sqrt{2a}$$

Prema tome $\iint_S\frac{dx\,dy}{\sqrt{2a-x}} = \frac{8a}{3}\sqrt{2a}$

(#)

Odrediti projekcije *l* linije *L* na ravan *xoy*:

77. $L: 4-x^2-y^2=z, \quad z=y^2.$

78. $L: x^2+y^2=z^2, \quad (z>0), \quad x+y+z=1.$

79. $L: x^2+y^2+z^2=4, \quad x^2+y^2=z^2, \quad (z>0).$

80. $L: x^2+y^2+z^2=a^2, \quad x+y+z=0.$

81. $L: 2x+y+z=1, \quad x-y-3z=5.$

82. $L: z=x^2+y^2, \quad z=2x+2y.$

Rješenja:

77. Linija L je data kao presjek površi $z=4-x^2-y^2$ i $z=y^2$ (paraboloid i cilindar). Projekcija linije L na ravan Oxy je skup onih tačaka (x, y) za koje je aplikata z sa jedne površi jednaka aplikati z sa druge površi, dakle, taj skup određujemo iz uslova

$$4-x^2-y^2=y^2.$$

Projekcija je, dakle, elipsa

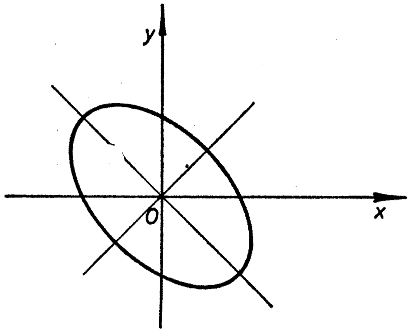
$$\frac{x^2}{2^2}+\frac{y^2}{(\sqrt{2})^2}=1.$$

78. $1=2x+2y-2xy.$

79. $x^2+y^2=2.$

80. Kriva ima projekciju

$$2x^2+2y^2+2xy-a^2=0.$$



Sl. 10

To je elipsa sa centrom u (0, 0). Ose elipse su prave $y=\pm x$, a poluose su a i $\frac{a}{3}$ (sl. 10).

81. $7x+2y-8=0.$ **82.** $(x-1)^2+(y-1)^2=2.$

(#)

Po definiciji izračunati integrale:

83. $\iint\limits_{\substack{0\leqslant x\leqslant a\\ 0\leqslant y\leqslant b}} xydxdy.$ **84.** $\iint\limits_{\substack{0\leqslant x\leqslant 1\\ 0\leqslant y\leqslant 1}} x^2ydxdy.$

Rješenja:

83. Funkcija $f(x, y)=xy$ je neprekidna pa i integrabilna na pravougaoniku $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$. Podijelimo dati pravougaonik pravama $x=x_i$ $(i=1, \dots, n)$, $y=y_j$ $(j=1, \dots, m)$. Po definiciji je

$$\iint\limits_{D} xydxdy=\lim_{\substack{m\to\infty\\ n\to\infty}}\sum_{i,j=1}^{n,m} f(M_{i,j})\cdot(x_i-x_{i-1})(y_j-y_{j-1})$$

pri čemu maksimalni podjeljak teži nuli kada $m\to\infty$, $n\to\infty$. Izaberimo da je:

$$x_i=\frac{a}{n}\cdot i, \qquad y_j=\frac{b}{m}j,$$

i da je

$$M_{i,j}=\left(\frac{a}{n}(i-1), \quad \frac{b}{m}(j-1)\right).$$

Biće:

$$\iint\limits_{\substack{0\leqslant x\leqslant a\\ 0\leqslant y\leqslant b}} xydxdy=\lim_{\substack{n\to\infty\\ m\to\infty}}\sum_{i=1}^{n}\sum_{j=1}^{m}\frac{a}{n}(i-1)\cdot\frac{b}{m}(j-1)\cdot\frac{ab}{n\cdot m}=$$

$$=\lim_{\substack{n\to\infty\\ m\to\infty}}\frac{a^2}{n^2}\cdot\frac{b^2}{m^2}\sum_{i=1}^{n}(i-1)\sum_{j=1}^{m}(j-1)=$$

$$=\lim_{\substack{n\to\infty\\ m\to\infty}}\frac{a^2b^2}{n^2\cdot m^2}\cdot\frac{(n-1)\cdot n}{2}\cdot\frac{(m-1)\cdot m}{2}=\frac{a^2b^2}{4}.$$

84. $\frac{1}{6}.$

Po definiciji izračunati integral:

$$\mathbf{85.}\ \iint\limits_{\substack{a\leqslant x\leqslant b\\ c\leqslant y\leqslant d}} e^{x+y}\,dxdy.$$

Rješenja:

85. Integral postoji jer je funkcija $f(x, y)=e^{x+y}$ neprekidna. Segment $[a, b]$ podijelimo pravama $x_i=a+\frac{b-a}{n}\cdot i$, $(i=0, 1, \dots, n)$, a $[c, d]$ pravama $y_j=c+\frac{d-c}{m}\cdot j$, $(j=0, 1, \dots, m)$, i u pravougaoniku $[x_{i-1}, x_i; y_{j-1}, y_j]$ uočimo tačku $M_{ij}=\left(a+\frac{b-a}{n}\cdot i,\ c+\frac{d-c}{m}\cdot j\right)$. Formirajmo integralnu sumu

$$S_{m,n}=\sum_{i,j=1}^{n,m} f(M_{i,j})(x_i-x_{i-1})(y_i-y_{i-1})=$$

$$=\sum_{i=1}^{n}\sum_{j=1}^{m} e^{a+\frac{b-a}{n}\cdot i}\cdot e^{c+\frac{d-c}{m}j}\cdot\frac{b-a}{n}\cdot\frac{d-c}{m}=$$

$$=e^a\cdot e^c\cdot\frac{b-a}{n}\cdot\frac{d-c}{m}\cdot\sum_{i=1}^{n} e^{\frac{b-a}{n}\cdot i}\cdot\sum_{j=1}^{m} e^{\frac{d-c}{m}j}.$$

Dobili smo geometrijske sume, pa je

$$S_{m,n}=e^a\cdot\frac{b-a}{n}\cdot e^{\frac{b-a}{n}}\cdot\frac{1-\left(e^{\frac{b-a}{n}}\right)^n}{1-e^{\frac{b-a}{n}}}\cdot e^c\cdot\frac{d-c}{m}\cdot e^{\frac{d-c}{m}}\cdot\frac{1-\left(e^{\frac{d-c}{m}}\right)^m}{1-e^{\frac{d-c}{m}}}=$$

$$=e^a\cdot\frac{b-a}{n}\cdot e^{\frac{b-a}{n}}\cdot\frac{1-e^{b-a}}{1-e^{\frac{b-a}{n}}}\cdot e^c\cdot\frac{d-c}{n}\cdot e^{\frac{d-c}{n}}\cdot\frac{1-e^{d-c}}{1-e^{\frac{d-c}{m}}}.$$

Kako je

$$\lim_{n\to\infty}\frac{b-a}{n}\cdot\frac{1}{1-e^{\frac{b-a}{n}}}=\lim_{\alpha\to 0}\frac{\alpha}{1-e^{\alpha}}=-1$$

i

$$\lim_{m\to\infty}\frac{d-c}{m}\,\frac{1}{1-e^{\frac{d-c}{m}}}=-1,$$

to

$$\lim_{\substack{m\to\infty\\ n\to\infty}} \mathrm{S}_{m,n}=e^a\cdot\frac{(1-e^{b-a})}{-1}\cdot e^c\cdot\frac{(1-e^{d-c})}{-1}=$$

$$=(e^a-e^b)\cdot(e^c-e^d).$$

Dakle,

$$\iint\limits_{\substack{a\leqslant x\leqslant b\\ c\leqslant y\leqslant d}} e^{x+y}\,dxdy=(e^a-e^b)(e^c-e^d).$$

Promijeniti poredak integracije u sljedećim integralima uzimajući da je $f(x, y)$ neprekidna funkcija:

$$\mathbf{86.}\ \int_0^1 dx\int_0^x f(x, y)\,dy. \qquad \mathbf{87.}\ \int_1^e dx\int_0^{\ln x} f(x, y)\,dy.$$

$$\mathbf{88.}\ \int_0^4 dx\int_{3x^2}^{12x} f(x, y)\,dy. \qquad \mathbf{89.}\ \int_0^1 dx\int_x^{2-x} f(x, y)\,dy.$$

$$\mathbf{90.}\ \int_0^1 dy\int_{\sqrt{y}}^{2-y} f(x, y)\,dx. \qquad \mathbf{91.}\ \int_{-7}^1 dx\int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}} f(x, y)\,dx.$$

Rješenja:

$$\mathbf{86.}\ \int_0^1 dy\int_y^1 f(x, y)\,dx. \qquad \mathbf{87.}\ \int_0^1 dy\int_{e^y}^{e} f(x, y)\,dx.$$

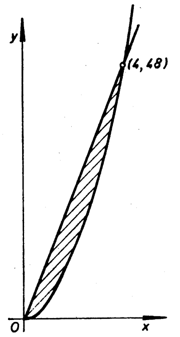
88. $x=12x\Rightarrow x=\frac{y}{12}$,

$$y=3x^2\Rightarrow x=\sqrt{\frac{y}{3}},$$

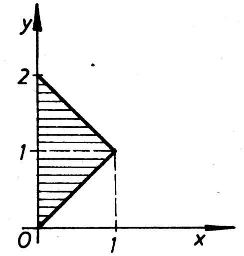
$$\int_0^{48} dy\int_{\frac{y}{12}}^{\sqrt{\frac{y}{3}}} f(x, y)\,dx \text{ (sl. 11).}$$

$$\mathbf{89.}\ \int_0^1 dy\int_0^y f(x, y)\,dx+$$

$$+\int_1^2 dy\int_0^{2-y} f(x, y)\,dx. \text{ (sl. 12).}$$



Sl. 11



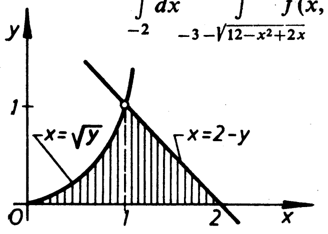
Sl. 12

90. Oblast integracije (sl. 13) dijelimo na dvije; biće

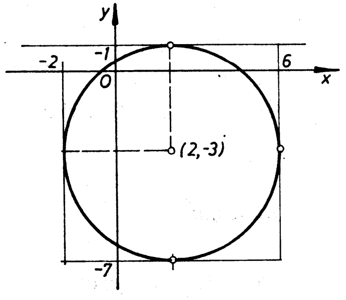
$$\int_1^2 dx\int_0^{x^2} f(x, y)\,dy+\int_1^2 dx\int_0^{2-x} f(x, y)\,dy.$$

91. Nacrtajmo linije koje ograničavaju oblast integracije $x_1=2-\sqrt{7-6y-y^2}$, $x_2=2+\sqrt{7-6y-y^2}$, $y=-7$, $y=1$. Linije x_1 i x_2 su polukružnice kružnice $(x-2)^2+(y+3)^2=16$. Oblast integracije je unutrašnjost kruga (sl. 14), pa je dati integral jednak integralu

$$\int_{-2}^{6} dx\int_{-3-\sqrt{12-x^2+2x}}^{-3+\sqrt{12-x^2+2x}} f(x, y)\,dy.$$



Sl. 13



Sl. 14

(#) Promijeniti poredak integracije u sljedećim integralima uzimajući da je $f(x, y)$ neprekidna funkcija:

92. $\int_0^1 dy \int_{2-y}^{1+\sqrt{1-y^2}} f(x, y)\, dx.$ **93.** $\int_0^{\pi} dx \int_0^{\sin x} f(x, y)\, dy.$

94. $\int_{-1}^0 dy \int_{-2\sqrt{1+y}}^{2\sqrt{1+y}} f(x, y)\, dx + \int_0^8 dy \int_{-2\sqrt{1+y}}^{2-y} f(x, y)\, dx.$

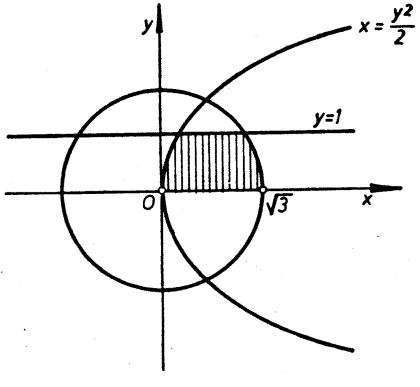
95. $\int_0^1 dx \int_{\frac{1}{2}(1-x^2)}^{\sqrt{1-x^2}} f(x, y)\, dy.$ **96.** $\int_0^1 dy \int_{\frac{y^2}{2}}^{\sqrt{3-y^2}} f(x, y)\, dx.$

Rješenja:

92. $\int_1^2 dx \int_{2-x}^{\sqrt{2x-x^2}} f(x, y)\, dy.$

93. $\int_0^1 dy \int_{\arcsin y}^{\pi - \arcsin y} f(x, y)\, dx.$ **94.** $\int_{-6}^2 dx \int_{\frac{x^2}{4}-1}^{2-x} f(x, y)\, dy.$

95. $\int_0^{1/2} dy \int_{\sqrt{1-2y}}^{\sqrt{1-y^2}} f(x, y)\, dx + \int_{1/2}^1 dy \int_0^{\sqrt{1-y^2}} f(x, y)\, dx.$



Sl. 15

96. Oblast integracije (sl. 15) ograničena je pravama $y=0$, $y=1$, lukom parabole $x=\frac{y^2}{2}$ i lukom kružnice $x^2+y^2=3$, $(x>0)$.

Prava $y=1$ i parabola sijeku se u tački sa apscisom $\frac{1}{2}$; prava $y=1$ i luk $x=\sqrt{3-y^2}$ sijeku se u tački sa apscisom $\sqrt{2}$. Otuda je

$$\int_0^{1/2} dx \int_0^{\sqrt{2x}} f(x, y)\, dy + \int_{1/2}^1 dx \int_0^1 f(x, y)\, dy +$$

$$+ \int_{\sqrt{2}}^{\sqrt{3}} dx \int_0^{\sqrt{3-x^2}} f(x, y)\, dy.$$

(#)

Izračunati integral I:

97. $\iint\limits_{\substack{0\leqslant x\leqslant\frac{\pi}{2}\\ 0\leqslant y\leqslant\frac{\pi}{2}}} \sin(x+y)\, dxdy.$ **98.** $\iint\limits_{\substack{-1\leqslant x\leqslant 1\\ -2\leqslant y\leqslant 2}} \left(1-\frac{1}{3}x-\frac{1}{4}y\right) dxdy.$

99. $\iint\limits_{\substack{0\leqslant x\leqslant 1\\ 0\leqslant y\leqslant x}} (x^2+y^2)\, dxdy.$

Rješenja:

97. Integral ćemo izračunati svođenjem na jednostruke integrale. Biće

$$I=\int_0^{\pi/2} dx \int_0^{\pi/2} \sin(x+y)\, dy = \int_0^{\pi/2} dx\,[-\cos(x+y)]\Big|_0^{\pi/2} = \int_0^{\pi/2}\left[\cos x - \cos\left(x+\frac{\pi}{2}\right)\right] dx =$$

$$= \sin x\Big|_0^{\pi/2} - \sin\left(x+\frac{\pi}{2}\right)\Big|_0^{\pi/2} = 1+1=2.$$

98. $$I=\int_{-1}^1 dx \int_{-2}^2 \left(1-\frac{1}{3}x-\frac{1}{4}y\right) dy = \int_{-1}^1 \left(y-\frac{1}{3}xy-\frac{1}{8}y^2\right)\Big|_{-2}^2 dx =$$

$$= \int_{-1}^1 \left(4-\frac{4}{3}x\right) dx = \left(4x-\frac{2}{3}x^2\right)\Big|_{-1}^1 = 8.$$

Ako se izvrši integracija najprije po x pa po y, dobiće se isti rezultat:

$$I=\int_{-2}^2 dy \int_{-1}^1 \left(1-\frac{1}{3}x-\frac{1}{3}y\right) dx = \int_{-2}^2 \left(x-\frac{1}{6}x^2-\frac{1}{4}xy\right)\Big|_{-1}^1 dy =$$

$$= \int_{-2}^2 \left(2-\frac{1}{2}y\right) dy = \left(2y-\frac{1}{4}y^2\right)\Big|_{-2}^2 = 8.$$

99. $$I=\int_0^1 dx \int_0^x (x^2+y^2)\, dy = \int_0^1 dx \left(x^2 y+\frac{y^3}{3}\right)\Big|_0^x = \int_0^1 \left(x^3+\frac{x^3}{3}\right) dx = \frac{1}{3}.$$

\# Izračunati integral *I*:

$$\textbf{100.}\quad \iint\limits_{\substack{-3\leqslant x\leqslant 1\\ 2x-1\leqslant y\leqslant 2-x^2}} (x-y)\,dxdy\,.$$

$$\textbf{101.}\quad \iint\limits_{\substack{0\leqslant x\leqslant 1\\ 0\leqslant y\leqslant 1-x}} (2x^2+y^2+1)\,dxdy. \qquad \textbf{102.}\quad \iint\limits_{|x|+|y|\leqslant 1} x^2\,dxdy.$$

103. $\iint\limits_D (x+y)\,dxdy$, gdje je D oblast ograničena linijama $y=x^2$, $y=x$.

Rješenja:

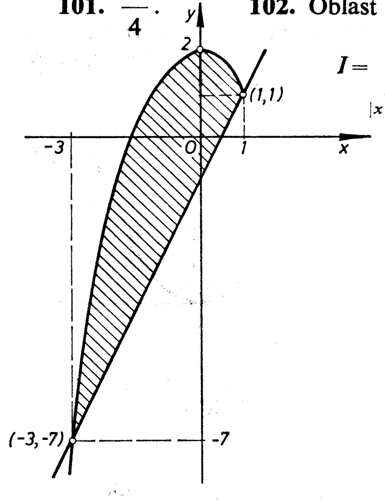
100. Prava $y=2x-1$ i parabola $y=2-x^2$ sijeku se u tačkama $(-3,\ -7)$, $(1,\ 1)$ (sl. 16). Biće:

$$I=\int\limits_{-3}^{1} dx\int\limits_{2x-1}^{2-x^2}(x-y)\,dy=\int\limits_{-3}^{1}\left(xy-\frac{y^2}{2}\right)\Bigg|_{2x-1}^{2-x^2}dx=$$

$$=\int\limits_{-3}^{1}\left(2x-x^3-2+2x^2-\frac{1}{2}x^4-2x^2+x+2x^2-2x+\frac{1}{2}\right)dx=$$

$$=\int\limits_{-3}^{1}\left(-\frac{1}{2}x^4-x^3+2x^2+x-\frac{3}{2}\right)dx=4\frac{4}{15}.$$

101. $\dfrac{3}{4}$.

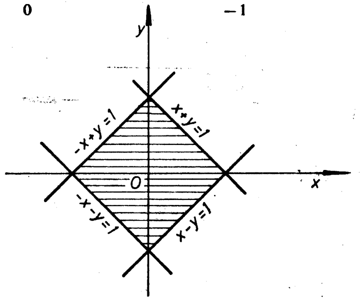


Sl. 16

102. Oblast integracije prikazana je na sl. 17. Biće:

$$I=\iint\limits_{|x|+|y|\leqslant 1} x^2\,dxdy=\int\limits_{0}^{1}x^2dx\int\limits_{x-1}^{1-x}dy+\int\limits_{-1}^{0}x^2dx\int\limits_{-1-x}^{1+x}dy=$$

$$=2\int\limits_{0}^{1}x^2(1-x)\,dx+2\int\limits_{-1}^{0}x^2(1+x)\,dx=\frac{1}{3}.$$



Sl. 17

$$\textbf{103.}\quad I=\int\limits_{0}^{1}dx\int\limits_{x^2}^{x}(x+y)\,dy=\int\limits_{0}^{1}\left(x^2+\frac{x^2}{2}-x^3-\frac{x^4}{2}\right)dx=\frac{3}{20}.$$

Zadaci za vježbu

U zadacima 3466 — 3476 proceniti date integrale.

3466. $\iint\limits_D (x+y+10)\,d\sigma$, gde je D—krug $x^2+y^2<4$.

3467. $\iint\limits_D (x^2+4y^2+9)\,d\sigma$, gde je D—krug $x^2+y^2<4$.

3468. $\iint\limits_D (x+y+1)\,d\sigma$, gde je D—pravougaonik $0<x<I$, $0<y<2$.

3469. $\iint\limits_D (x+xy-x^2-y^2)\,d\sigma$, gde je D—pravougaonik $0<x<1$, $0<y<2$.

3470. $\iint\limits_D xy(x+y)\,d\sigma$, gde je D—kvadrat $0\leqslant x\leqslant 2$, $0<y<2$.

3471. $\iint\limits_D (x+1)^y\,d\sigma$, gde je D—kvadrat $0\leqslant x\leqslant 2$, $0<y\leqslant 2$.

3472. $\iint\limits_D (x^2+y^2-2\sqrt{x^2+y^2}+2)\,d\sigma$, gde je D—kvadrat $0<x<2$, $0<y<2$.

3473. $\iint\limits_D (x^2+y^2-4x-4y+10)\,d\sigma$, gde je D—oblast ograničena elipsom $x^2+4y^2-2x-16y+13=0$ (uključujući granicu).

U zadacima 3477 — 3484 izračunati date dvojne integrale po pravougaonim oblastima D koje su određene nejednakostima navedenim u zadacima.

3477. $\iint\limits_D xy\,dx\,dy \quad (0<x<1,\ 0<y<2)$.

3478. $\iint\limits_D e^{x+y}\,dx\,dy \quad (0<x<1,\ 0<y<1)$.

3479. $\displaystyle\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy \quad (0<x<1,\ 0<y<1)$.

3480. $\displaystyle\iint\limits_D \frac{dx\,dy}{(x+y+1)^2} \quad (0<x<1.\ 0\leqslant y<1)$.

3481. $\displaystyle\iint\limits_D \frac{y\,dx\,dy}{(1+x^2+y^2)^{\frac{3}{2}}} \quad (0<x<1,\ 0<y<1)$.

3482. $\displaystyle\iint\limits_D x\sin(x+y)\,dx\,dy \quad \left(0<x<\pi,\ 0\leqslant y<\frac{\pi}{2}\right)$.

3483. $\iint\limits_D x^2ye^{xy}\,dx\,dy \quad (0\leqslant x\leqslant 1,\ 0<y<2)$.

3484. $\displaystyle\iint\limits_D x^2y\cos(xy^2)\,dx\,dy \quad \left(0<x<\frac{\pi}{2},\ 0<y<2\right)$

U zadacima 3485 — 3497 naći granice dvcstrukcg integrala na koji se svodi dvojni integral $\iint\limits_D f(x,\ y)\,dx\,dy$ za date (konačne) oblasti integracije D.

3485. Paralelogram koji obrazuju prave

$$x=3,\quad x=5,\quad 3x-2y+4=0,\quad 3x-2y+1=0.$$

3486. Trougao koji obrazuju prave $x=0$, $y=0$, $x+y=2$.

3487. $x^2+y^2\leqslant 1$, $\quad x\geqslant 0$, $\quad y\geqslant 0$.

3488. $x+y\leqslant 1$, $\quad x-y\leqslant 1$, $\quad x\geqslant 0$.

3489. $y\geqslant x^2 \quad y\leqslant 4-x^2$.

3490. $\dfrac{x^2}{4}+\dfrac{y^2}{9}\leqslant 1$. $\qquad$ **3491.** $(x-2)^2+(y-3)^2\leqslant 4$.

Rješenja

3466. $8\pi(45-\sqrt{2})<I<8\pi(5+\sqrt{2})$.

3467. $36\pi<I<100\pi$.

3468. $2<I<8$. $\quad$ **3469.** $-8<I<\dfrac{2}{3}$.

2470. $0<I<64$.

3471. $4<I<36$.

3472. $4<I<8(5-2\sqrt{2})$. $\quad 4\pi<I<22\pi$.

3474. $0<I<\dfrac{4}{3}\pi R^5$.

3475. $24<I<72$.

3476. $29\pi\sqrt{3}<I<52\pi\sqrt{3}$.

3477. 1. $\quad$ **3478.** $(e-1)^2$.

3479. $\dfrac{\pi}{12}$.

3480. $\ln\dfrac{4}{3}$. $\quad$ **3481.** $\ln\dfrac{2+\sqrt{2}}{1+\sqrt{3}}$.

3482. $\pi-2$. $\quad$ **3483.** 2. $\quad$ **3484.** $-\dfrac{\pi}{16}$.

$$\textbf{3485.}\ \int\limits_3^5 dx\int\limits_{\frac{3x+1}{2}}^{\frac{3x+4}{2}} f(x,y)\,dy. \qquad \textbf{3486.}\ \int\limits_0^2 dx\int\limits_0^{2-x} f(x,y)\,dy.$$

$$\textbf{3487.}\ \int\limits_0^1 dx\int\limits_0^{\sqrt{1-x^2}} f(x,y)\,dy. \qquad \textbf{3488.}\ \int\limits_0^1 dx\int\limits_{x-1}^{1-x} f(x,y)\,dy.$$

$$\textbf{3489.}\ \int\limits_{-\sqrt{2}}^{\sqrt{2}} dx\int\limits_{x^2}^{4-x^2} f(x,y)\,dy. \qquad \textbf{3490.}\ \int\limits_{-2}^{2} dx\int\limits_{-\frac{3}{2}\sqrt{4-x^2}}^{\frac{3}{2}\sqrt{4-x^2}} f(x,y)\,dy.$$

$$\textbf{3491.}\ \int\limits_0^4 dx\int\limits_{3-\sqrt{4x-x^2}}^{3+\sqrt{4x-x^2}} f(x,y)\,dy.$$

3492. Oblast D je ograničena parabolama $y=x^2$ i $y=\sqrt{x}$.

3493. Trougao koji obrazuju prave $y=x$, $y=2x$ i $x+y=6$.

3494. Paralelogram koji obrazuju prave

$$y=x,\quad y=x+3,\quad y=-2x+1,\quad y=-2x+5.$$

3495. $y-2x<0$, $2y-x>0$, $xy<2$.

3496. $y^2<8x$, $y\leq 2x$, $y+4x-24<0$.

3497. Oblast D je ograničena hiperbolom $y^2-x^2=1$ i krugom $x^2+y^2=9$ (misli se na onu oblast u kojoj leži koordinatni početak).

U zadacima 3498 — 3503 promeniti redosled integracije u datim integralima.

3498. $\int_{0}^{1} dy \int_{y}^{\sqrt{y}} f(x, y)\, dx.$ **3499.** $\int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} f(x, y)\, dy.$

3500. $\int_{0}^{r} dx \int_{x}^{\sqrt{2rx-x^2}} f(x, y)\, dy.$

3501. $\int_{-2}^{2} dx \int_{-\frac{1}{\sqrt{2}}\sqrt{4-x^2}}^{\frac{1}{\sqrt{2}}\sqrt{4-x^2}} f(x, y)\, dy.$

3502. $\int_{1}^{2} dx \int_{x}^{2x} f(x, y)\, dy.$ **3503.** $\int_{0}^{2} dx \int_{2x}^{6-x} f(x, y)\, dy.$

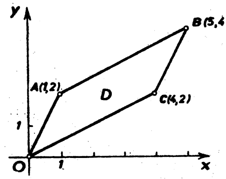
3504. Izmenivši redosled integracije predstaviti dati izraz u obliku jednog dvostrukog integrala:

1) $\int_{0}^{1} dx \int_{0}^{x} f(x, y)\, dy + \int_{1}^{2} dx \int_{0}^{2-x} f(x, y)\, dy;$

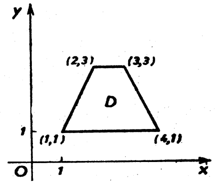
2) $\int_{0}^{1} dx \int_{0}^{x^2} f(x, y)\, dy + \int_{1}^{3} dx \int_{0}^{\frac{1}{2}(3-x)} f(x, y)\, dy;$

3) $\int_{0}^{1} dx \int_{0}^{x^{\frac{2}{3}}} f(x, y)\, dy + \int_{1}^{2} dx \int_{0}^{1-\sqrt{4x-x^2-3}} f(x, y)\, dy.$

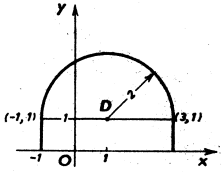
3505. Predstaviti dvojni integral $\iint\limits_{D} f(x, y)\, dx\, dy$ po oblastima D prikazanim na sl. 62, 63, 64, 65, — u obliku zbira dvostrukih integrala (sa najmanjim brojem sabiraka). Granične linije oblasti prikazanih na sl. 64 i 65 sastoje se iz pravolinijskih odsečaka i kružnih lukova.



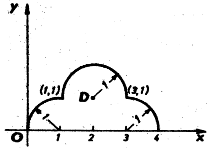
Sl. 62



Sl. 63



Sl. 64



Sl. 65

Rješenja

3492. $\int_{0}^{1} dx \int_{x^2}^{\sqrt{x}} f(x, y)\, dy.$

3493. $\int_{0}^{2} dx \int_{x}^{2x} f(x, y)\, dy + \int_{2}^{3} dx \int_{x}^{6-x} f(x, y)\, dy.$

3494. $\int_{-\frac{2}{3}}^{\frac{1}{3}} dx \int_{1-2x}^{x+3} f(x, y)\, dy + \int_{\frac{1}{3}}^{\frac{2}{3}} dx \int_{x}^{x+3} f(x, y)\, dy + \int_{\frac{2}{3}}^{\frac{5}{3}} dx \int_{x}^{5-2x} f(x, y)\, dy.$

3495. $\int_{0}^{1} dx \int_{\frac{x}{2}}^{2x} f(x, y)\, dy + \int_{1}^{2} dx \int_{\frac{x}{2}}^{\frac{2}{x}} f(x, y)\, dy.$

3496. $\int_{0}^{2} dx \int_{-2\sqrt{2x}}^{2x} f(x, y)\, dy + \int_{2}^{\frac{9}{2}} dx \int_{-2\sqrt{2x}}^{2\sqrt{2x}} f(x, y)\, dy + \int_{\frac{9}{2}}^{8} dx \int_{-2\sqrt{2x}}^{24-4x} f(x, y)\, dy.$

3497. $\int_{-3}^{-2} dy \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\, dx + \int_{-2}^{2} dx \int_{-\sqrt{1+x^2}}^{\sqrt{1+x^2}} f(x, y)\, dy + \int_{2}^{3} dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x, y)\, dy.$

3498. $\int_{0}^{1} dx \int_{x^2}^{x} f(x, y)\, dy.$ **3499.** $\int_{0}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x, y)\, dx.$

3500. $\int_{0}^{r} dy \int_{r-\sqrt{r^2-y^2}}^{y} f(x, y)\, dx.$

3501. $\int_{-\sqrt{2}}^{\sqrt{2}} dy \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} f(x, y)\, dx.$

3502. $\int_{1}^{2} dy \int_{1}^{y} f(x, y)\, dx + \int_{2}^{4} dy \int_{\frac{y}{2}}^{2} f(x, y)\, dx.$

3503. $\int_{0}^{4} dy \int_{0}^{\frac{y}{2}} f(x, y)\, dx + \int_{4}^{6} dy \int_{0}^{6-y} f(x, y)\, dx.$

3504. 1) $\int_{0}^{1} dy \int_{y}^{2-y} f(x, y)\, dx;$ 2) $\int_{0}^{1} dy \int_{\sqrt{y}}^{3-2y} f(x, y)\, dx;$

3) $\int_{0}^{1} dy \int_{y^{\frac{3}{2}}}^{2-\sqrt{2y-y^2}} f(x, y)\, dx.$

3505. 1) $\int_{0}^{2} dy \int_{\frac{y}{2}}^{2y} f(x, y)\, dx + \int_{2}^{4} dy \int_{2y-3}^{\frac{y+6}{2}} f(x, y)\, dx;$

2) $\int_{1}^{3} dy \int_{\frac{y+1}{2}}^{\frac{9-y}{2}} f(x, y)\, dx;$ 3) $\int_{-1}^{3} dx \int_{0}^{1+\sqrt{3+2x-x^2}} f(x, y)\, dy;$

4) $\int_{0}^{1} dy \int_{1-\sqrt{1-y^2}}^{3+\sqrt{1-y^2}} f(x, y)\, dx + \int_{1}^{2} dy \int_{2-\sqrt{2y-y^2}}^{2+\sqrt{2y-y^2}} f(x, y)\, dx.$

U zadacima 3506 — 3512 izračunati date integrale:

3506. 1) $\int_{0}^{a} dx \int_{0}^{\sqrt{x}} dy;$ 2) $\int_{2}^{4} dx \int_{x}^{2x} \frac{y}{x}\, dy;$ 3) $\int_{1}^{2} dy \int_{0}^{\ln y} e^x\, dx.$

3507. $\iint\limits_{D} x^3 y^2\, dx\, dy$, D—krug $x^2+y^2<R^2$.

3508. $\iint\limits_{D} (x^2+y)\, dx\, dy$, oblast D je ograničena parabolama $y=x^2$ i $y^2=x$.

3509. $\iint\limits_{D} \frac{x^2}{y^2}\, dx\, dy$, oblast D je ograničena pravama $x=2$, $y=x$ i hiperbolom $xy=1$.

3510. $\iint\limits_{D} \cos(x+y)\, dx\, dy$, oblast D je ograničena pravama $x=0$, $y=\pi$ i $y=x$.

3511. $\iint\limits_{D} \sqrt{1-x^2-y^2}$ oblast D je četvrtina kruga $x^2+y^2<1$, koji leži u prvom kvadrantu.

3512. $\iint\limits_{D} x^2 y^2 \sqrt{1-x^3-y^3}\, dx\, dy$, oblast D je ograničena krivom $x^3+y^3=1$ i koordinatnim osama.

3513. Naći srednju vrednost funkcije $z=12-2x-3y$ u oblasti ograničenoj pravama $12-2x-3y=0$, $x=0$, $y=0$.

3514. Naći srednju vrednost funkcije $z=2x+y$ u oblasi ograničenoj pravom $x+y=3$ i koordinatnim osama.

3515. Naći srednju vrednost funkcije $z=x+6y$ u oblasti ograničenoj pravama $y=x$, $y=5x$ i $x=1$.

3516. Naći srednju vrednost funkcije $z=\sqrt{R^2-x^2-y^2}$ u krugu $x^2+y^2<R^2$.

Rješenja

3506. 1) $\frac{2}{3}a^{\frac{3}{2}}$; 2) 9; 3) $\frac{1}{2}$. **3507.** 0. **3508.** $\frac{33}{140}$. **3509.** $\frac{9}{4}$.

3510. —2. **3511,** $\frac{\pi}{6}$. **3512.** $\frac{4}{135}$. **3513.** 4. **3514.** 3. **3515.** $12\frac{2}{3}$.

3516. $\frac{2}{3}R$.

Smjena promjenjivih u dvostrukom integralu

Neka je dat integral $I=\iint\limits_{D} f(x,y)\,dx\,dy$.

Ako uvodimo nove promjenjive u i v takve da je

$x=\varphi(u,v)$

$y=\psi(u,v)$ tada se oblast D preslikava u D'. Jakobijan transformacijom $J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}$ i imamo

$$I=\iint\limits_{D'} f(\varphi(u,v),\ \psi(u,v))\ |J|\,du\,dv,$$

$$dx\,dy=|J|\,du\,dv$$

Npr. smjena polarnim koordinatama izgleda

$x=r\cos\varphi$

$y=r\sin\varphi$, r i φ su polarne koordinate, $r\geq 0$

$0\leq\varphi\leq 2\pi$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r\cos^2\varphi+r\sin^2\varphi=r(\sin^2\varphi+\cos^2\varphi|=r$$

$$\iint\limits_{D} f(x,y)\,dx\,dy=\iint\limits_{D'} f(r\cos\varphi,\ r\sin\varphi)\cdot|r|\,d\varphi\,dr$$

Polarne koordinate obično uvodimo ako se u podintegralnoj f-ji ili u jednačinama koje opisuju oblast integracije pojavljuje izraz x^2+y^2.

Poopštene polarne koordinate izgledaju $x=ar\cos\varphi$ $(a>0)$

$y=br\sin\varphi$ $(b>0)$

$J=\dots=abr$ (ostavljamo za vježbu kako doći do ovog rezultata)

Ⓘ

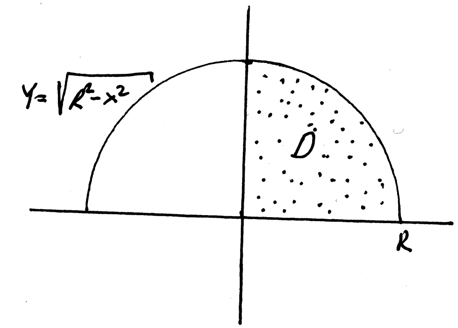
Dati dvostruki integral $\int\limits_0^R dx\int\limits_0^{\sqrt{R^2-x^2}} f(x,y)\,dy$

iz pravougaonih koordinata transformisati na polarne koordinate.

Rj. Oblast integracije D prema postavci zadatka je

$$D:\begin{cases}0\leq x\leq R\\ 0\leq y\leq\sqrt{R^2-x^2}\end{cases}$$

Skicirajmo oblast D.



$y^2=R^2-x^2$

$x^2+y^2=R^2$

Polarne koordinate glase

$x=r\cos\varphi$

$y=r\sin\varphi$

$dx\,dy=r\,dr\,d\varphi$

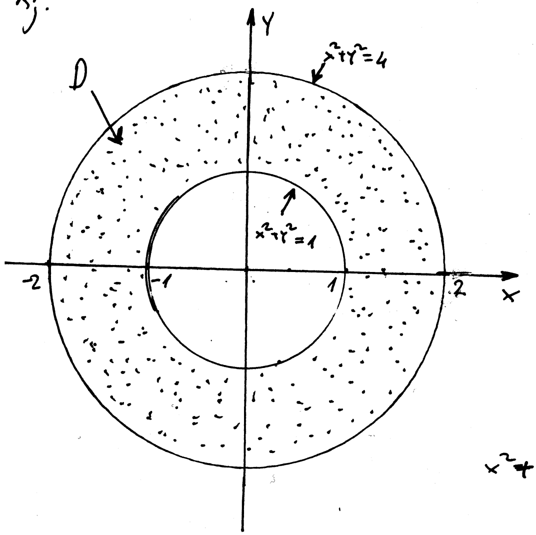
$$D\xrightarrow{\text{transformise}}D':\begin{cases}0\leq r\leq R\\ 0\leq\varphi\leq\frac{\pi}{2}\end{cases}$$

$$\int\limits_0^R dx\int\limits_0^{\sqrt{R^2-x^2}} f(x,y)\,dy=\int\limits_0^R r\,dr\int\limits_0^{\pi/2} f(r\cos\varphi,\ r\sin\varphi)\,d\varphi$$

(#) Izračunati dvostruki integral $I=\iint\limits_D \frac{dxdy}{\sqrt{x^2+y^2}}$ gdje je

D-kružni kolut, oblast omeđen krugovima $x^2+y^2=1$ i $x^2+y^2=4$ (drugim riječima $D=\{(x,y) \mid x,y\in\mathbb{R} \; i \; 1\leq x^2+y^2\leq 4\}$.

Rj.



Zadatak ćemo riješiti prelaskom na polarne koordinate

$x=r\cos\varphi$

$y=r\sin\varphi$

$dxdy=rdrd\varphi$

$D \xrightarrow{\text{transformacije}} D'$

gdje je $D'=\begin{cases} 1\leq r\leq 2 \\ 0\leq\varphi\leq 2\pi \end{cases}$

$x^2+y^2=r^2\cos^2\varphi+r^2\sin^2\varphi=r^2$

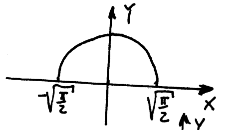
$$\iint\limits_D \frac{dxdy}{\sqrt{x^2+y^2}} = \iint\limits_{D'} \frac{rdrd\varphi}{\sqrt{r^2}} = \iint\limits_{D'} drd\varphi = \int\limits_1^2 dr \int\limits_0^{2\pi} d\varphi = 2\pi\cdot 1 = 2\pi$$

(#) Izračunati dvostruki integral

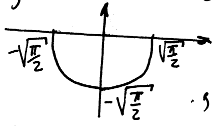
$$I=\int\limits_0^{\sqrt{\frac{\pi}{2}}} dx \int\limits_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}} \cos(x^2+y^2)\,dy \, .$$

Rj. Oblast integracije D je $D:\begin{cases} 0\leq x\leq\sqrt{\frac{\pi}{2}} \\ -\sqrt{\frac{\pi}{2}-x^2}\leq y\leq\sqrt{\frac{\pi}{2}-x^2} \end{cases}$

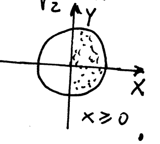
Znamo da f-ja $y=\sqrt{\frac{\pi}{2}-x^2}$ izgleda ovako



dok f-ja $y=-\sqrt{\frac{\pi}{2}-x^2}$ izgleda

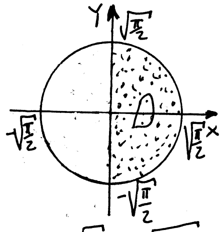


, sve zajedno $\Rightarrow$



$x\geq 0$.

Ove dvije f-je se dobiju iz f-je $y^2=\frac{\pi}{2}-x^2$ tj. $x^2+y^2=\frac{\pi}{2}$ što predstavlja jednačinu kruga sa centrom u koordinatnom početku, poluprečnika $\sqrt{\frac{\pi}{2}}$.



Uvedimo polarne koordinate

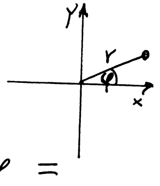
$x=r\cos\varphi$

$y=r\sin\varphi$

$dxdy=rdrd\varphi$

$D \xrightarrow{\text{transform.}} D':\begin{cases} 0\leq r\leq\sqrt{\frac{\pi}{2}} \\ -\frac{\pi}{2}\leq\varphi\leq\frac{\pi}{2} \end{cases}$

$x^2+y^2=\ldots=r^2$

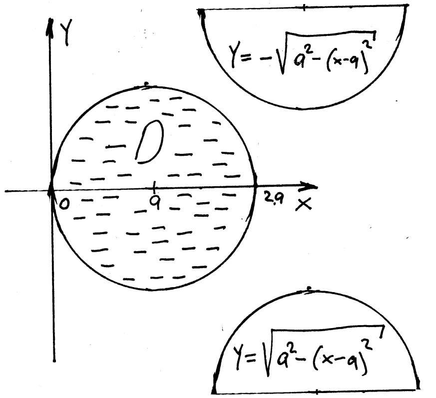


$$I=\int\limits_0^{\sqrt{\frac{\pi}{2}}} dx \int\limits_{-\sqrt{\frac{\pi}{2}-x^2}}^{\sqrt{\frac{\pi}{2}-x^2}} \cos(x^2+y^2)\,dy = \iint\limits_D \cos(x^2+y^2)dxdy = \iint\limits_{D'} \cos(r^2)\,rdrd\varphi =$$

$$= \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int\limits_0^{\sqrt{\frac{\pi}{2}}} r\cos(r^2)\,dr = \begin{vmatrix} r^2=t \\ 2rdr=dt \\ rdr=\frac{1}{2}dt \\ r\Big|_0^{\sqrt{\frac{\pi}{2}}} \Rightarrow t\Big|_0^{\frac{\pi}{2}} \end{vmatrix} = \frac{1}{2}\int\limits_{-\pi/2}^{\pi/2} d\varphi \int\limits_0^{\pi/2} \cos t\,dt = \frac{1}{2}\cdot\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cdot\sin t\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\cdot\pi\cdot 1 = \frac{\pi}{2}$$ traženo rješenje

(#) Izračunati $\iint\limits_D (x^2+y^2)\,dx\,dy$ gdje je D unutrašnjost kruga $x^2+y^2=2ax$.



Rj.

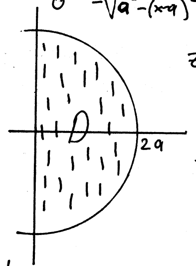
$$x^2+y^2=2ax$$
$$x^2-2ax+y^2=0$$
$$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$$
$$(x-a)^2+y^2=a^2$$

$S(a,0)$ centar

poluprečnik a

$$\iint\limits_D (x^2+y^2)\,dx\,dy=\int_0^{2a}\left(\int_{-\sqrt{a^2-(x-a)^2}}^{\sqrt{a^2-(x-a)^2}}(x^2+y^2)\,dy\right)dx=$$

$$\left|\begin{array}{l} x=r\cos\varphi \\ y=r\sin\varphi \\ x^2+y^2=r^2 \end{array}\right.$$

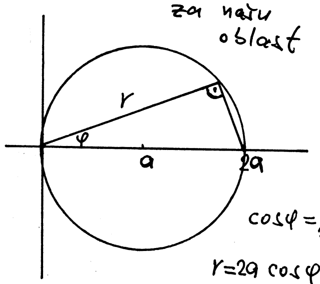


za ovakvu oblast imali bi

$0\le r\le 2a$

$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$

za našu oblast



$\cos\varphi=\frac{r}{2a}$

$r=2a\cos\varphi$

$0\le r\le 2a\cos\varphi$

$-\frac{\pi}{2}\le\varphi\le\frac{\pi}{2}$

$$dx\,dy=|J|\,dr\,d\varphi$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r \quad\Bigg| =$$

$$=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2a\cos\varphi}r^2|r|\,dr\right]d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[\int_0^{2a\cos\varphi}r^3\,dr\right]d\varphi=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{4}r^4\Big|_0^{2a\cos\varphi}d\varphi=4a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^4\varphi\,d\varphi$$

$$=\left|\begin{array}{l}\cos^4\varphi=(\cos^2\varphi)^2=\left(\frac{1+\cos2\varphi}{2}\right)^2=\frac{1}{4}(\cos^2 2\varphi+2\cos2\varphi+1)\\ 1=\sin^2\varphi+\cos^2\varphi\\ \cos2\varphi=\cos^2\varphi-\sin^2\varphi\end{array}\right|=$$

$$=a^4\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos^2 2\varphi+2\cos2\varphi+1)\,d\varphi=a^4\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{2}(1+\cos4\varphi)\,d\varphi+2\cdot\frac{1}{2}\sin2\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\right]\overset{(*)}{=}$$

$$\left|\begin{array}{l}1=\sin^2 2\varphi+\cos^2 2\varphi\\ \cos4\varphi=\cos^2 2\varphi-\sin^2\varphi^2\end{array}\right.$$

$$1+\cos4\varphi=2\cos^2 2\varphi$$
$$\cos^2 2\varphi=\frac{1}{2}(1+\cos4\varphi)$$

$$\int\cos2\varphi\,d\varphi=\begin{vmatrix}2\varphi=t\\ 2d\varphi=dt\\ d\varphi=\frac{1}{2}dt\end{vmatrix}=\frac{1}{2}\int\cos t\,dt$$
$$=\frac{1}{2}\sin t+c=\frac{1}{2}\sin2\varphi+c$$

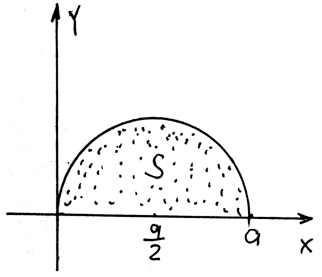
$$\overset{(*)}{=}a^4\left[\frac{1}{2}\pi+\frac{1}{2}\cdot\frac{1}{4}\sin4\varphi\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}+0+\pi\right]=a^4\left[\frac{3\pi}{2}+\frac{1}{8}\cdot0\right]=\frac{3\pi}{2}a^4$$

II način: Uvodimo smjenu $x=a+r\cos\varphi$, $y=r\sin\varphi$, $0\le\varphi\le2\pi$, $0\le r\le a$

URADITI ZA VJEŽBU

Ⓧ Izračunati integral $\iint_S y\,dx\,dy$ gdje je S unutrašnjost gornjeg polukruga poluprečnika $\frac{a}{2}$ sa središtom u tački $(\frac{a}{2},0)$.

Rj.



I način:

$$\iint_S y\,dx\,dy = \dots = \int_0^{\frac{\pi}{2}}\left[\int_0^{a\cos\varphi} r\sin\varphi\cdot|r|\,dr\right]d\varphi$$

$$= \dots = \frac{a^3}{12}$$

OSTAVJAMO ZA VJEŽBU KAKO SMO OVO DOBILI

$dx\,dy = |J|\,dr\,d\varphi$

II način:

$$\iint_S y\,dx\,dy = \left|\begin{array}{l} x = \frac{a}{2} + r\cos\varphi \\ y = r\sin\varphi \\ 0 \le \varphi \le \pi \\ 0 \le r \le \frac{a}{2} \end{array}\right.$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}$$

$$J = r$$

$$= \int_0^{\pi}\left[\int_0^{\frac{a}{2}} r\sin\varphi\cdot|r|\,dr\right]d\varphi = \int_0^{\pi}\sin\varphi\,\frac{1}{3}r^3\Big|_0^{\frac{a}{2}}\,d\varphi = \frac{a^3}{24}\int_0^{\pi}\sin\varphi\,d\varphi =$$

$$= \frac{a^3}{24}(-\cos\varphi)\Big|_0^{\pi} = -\frac{a^3}{24}(-1-1) = \frac{a^3}{12}$$

Ⓧ Izračunati dvostruki integral dat u polarnim koordinatama $I = \iint_D \rho\sin\varphi\,d\rho\,d\varphi$ gdje je oblast D

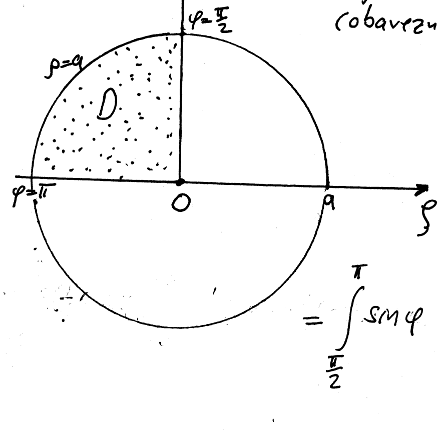
a) kružni sektor, ograničen linijama $\rho = a$, $\varphi = \frac{\pi}{2}$ i $\varphi = \pi$

b) polukrug $\rho \le 2a\cos\varphi$, $0 \le \varphi \le \frac{\pi}{2}$

c) oblast između linija $\rho = 2 + \cos\varphi$ i $\rho = 1$.

(obavezno nacrtati izgled oblasti D)

Rj. a)



$$I = \iint_D \rho\sin\varphi\,d\rho\,d\varphi =$$

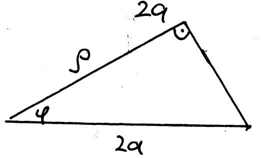
$$= \int_{\frac{\pi}{2}}^{\pi}\sin\varphi\,d\varphi\int_0^{a}\rho\,d\rho =$$

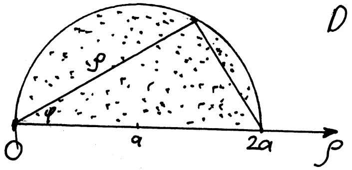
$$= \int_{\frac{\pi}{2}}^{\pi}\sin\varphi\,\frac{\rho^2}{2}\Big|_0^{a}\,d\varphi = \frac{a^2}{2}\left(-\cos\varphi\Big|_{\frac{\pi}{2}}^{\pi}\right) = \frac{a^2}{2}$$

b)

$\rho = 2a\cos\varphi$

$\cos\varphi = \frac{\rho}{2a}$





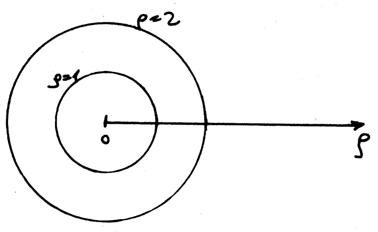
$$D: \begin{cases} 0 \le \rho \le 2a\cos\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \end{cases}$$

$$I = \iint_D \rho\sin\varphi\,d\rho\,d\varphi = \int_0^{\frac{\pi}{2}}\sin\varphi\,d\varphi\int_0^{2a\cos\varphi}\rho\,d\rho = \int_0^{\frac{\pi}{2}}\frac{1}{2}\rho^2\Big|_0^{2a\cos\varphi}\sin\varphi\,d\varphi =$$

$$= \frac{1}{2}\cdot 4a^2\int_0^{\pi/2}\sin\varphi\cos^2\varphi\,d\varphi = \left|\begin{array}{l} \cos\varphi = t \\ -\sin\varphi\,d\varphi = dt \\ \varphi\Big|_0^{\frac{\pi}{2}} \Rightarrow t\Big|_1^{0} \end{array}\right| = 2a^2\left(-\int_1^{0} t^2\,dt\right) =$$

$$= 2a^2\int_0^{1} t^2\,dt = 2a^2\cdot\frac{t^3}{3}\Big|_0^{1} = \frac{2}{3}a^2$$ traženo rješenje

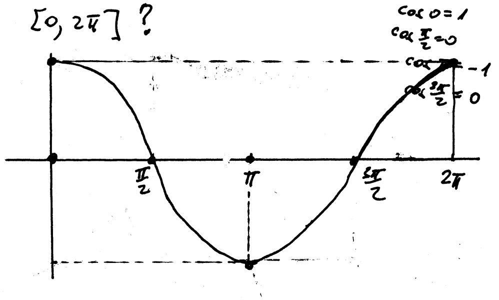
c) linije $\rho=1$ i $\rho=2$ nije teško nacrtati



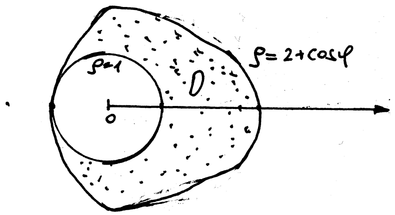
Problem predstavlja linija

$\rho = 2+\cos\varphi$

Kako izgleda $\cos\varphi$ na intervalu $[0, 2\pi]$?



Ako liniji $\rho=2$ dodamo $\cos\varphi$ imamo otprilike sljedeću sliku:



$$D: \begin{cases} 1 \le \rho \le 2+\cos\varphi \\ 0 \le \varphi \le 2\pi \end{cases}$$

$$I=\iint\limits_D \rho\sin\varphi\, d\rho\, d\varphi = \int_0^{2\pi}\sin\varphi\, d\varphi \int_1^{2+\cos\varphi}\rho\, d\rho = \int_0^{2\pi}\frac{\rho^2}{2}\Big|_1^{2+\cos\varphi}\sin\varphi\, d\varphi =$$

$$=\frac{1}{2}\int_0^{2\pi}\left((2+\cos\varphi)^2-1^2\right)\sin\varphi\, d\varphi = \frac{1}{2}\int_0^{2\pi}(4+4\cos\varphi+\cos^2\varphi-1)\sin\varphi\, d\varphi$$

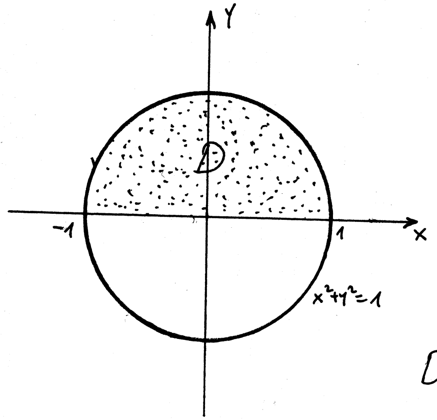
$$=-\frac{1}{2}\int_0^{2\pi}(\cos^2\varphi+4\cos\varphi+3)\, d\cos\varphi = \left(-\frac{1}{2}\right)\left(\frac{\cos^3\varphi}{3}\Big|_0^{2\pi}+4\,\frac{\cos^2\varphi}{2}\Big|_0^{2\pi}+3\cos\varphi\Big|_0^{2\pi}\right)$$

$$=\left(-\frac{1}{2}\right)\left(\frac{1}{3}(1-1)+2(1-1)+3(1-1)\right)=0$$ traženo rješenje

---

Ⓗ Izračunati integral $I=\iint\limits_D\sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$, ako je D oblast data sa: $x^2+y^2\le 1$, $y\ge 0$.

Rj. Skicirajmo oblast D



$$D: \begin{cases} -1 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{cases}$$

Uvedimo polarne koordinate

$x = r\cos\varphi$

$y = r\sin\varphi$

$dx\,dy = r\,dr\,d\varphi$

$$D \xrightarrow{\text{transformише}} D', \quad D': \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \pi \end{cases}$$

$1-x^2-y^2 = 1-(x^2+y^2) = 1-r^2$

$1+x^2+y^2 = 1+r^2$

$$I=\iint\limits_D\sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy = \iint\limits_{D'}\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr\,d\varphi = \int_0^{\pi}d\varphi\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr$$

Izračunajmo posebno drugi integral

$$\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr = \int_0^1\frac{\sqrt{1-r^2}}{\sqrt{1+r^2}}\,r\,dr = \int_0^1\frac{1-r^2}{\sqrt{(1+r^2)(1-r^2)}}\cdot r\,dr = \int_0^1\frac{r}{\sqrt{1-r^4}}\,dr - \int_0^1\frac{r^3}{\sqrt{1-r^4}}\,dr$$

$r|_0^1 \Rightarrow t|_0^1$

$$\int_0^1\frac{r}{\sqrt{1-r^4}}\,dr = \left|\begin{array}{l} r^2=t \\ 2r\,dr=dt \\ r\,dr=\frac{1}{2}dt \end{array}\right| = \frac{1}{2}\int_0^1\frac{dt}{\sqrt{1-t^2}} = \frac{1}{2}\arcsin t\Big|_0^1 = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}$$

$$\int_0^1\frac{r^3}{\sqrt{1-r^4}}\,dr = \left|\begin{array}{ll} 1-r^4=s^2 & r^3dr=-\frac{1}{2}s\,ds \\ -4r^3dr=2s\,ds & r|_0^1 \Rightarrow s|_1^0 \end{array}\right| = -\frac{1}{2}\int_1^0\frac{s\,ds}{\sqrt{s^2}} = \frac{1}{2}$$

$$I=\int_0^{\pi}d\varphi\int_0^1\sqrt{\frac{1-r^2}{1+r^2}}\,r\,dr = \varphi\Big|_0^{\pi}\cdot\left(\frac{\pi}{4}-\frac{1}{2}\right) = \frac{\pi^2}{4}-\frac{\pi}{2}$$ traženo rješenje

\# Izračunati integral $I=\int\limits_{0}^{\sqrt{3}/2} dx \int\limits_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy$.

Rj. Pokušajmo prvo skicirati oblast integracije D. Primjetimo da se u drugom integralu pojavljuju f-je $y=\sqrt{1-x^2}$ i $y=1-\sqrt{1-x^2}$. Nacrtajmo ih.

$y=\sqrt{1-x^2}$

$y^2=1-x^2$

$x^2+y^2=1$

krug sa centrom
u $C(0,0)$
poluprečnika $r=1$

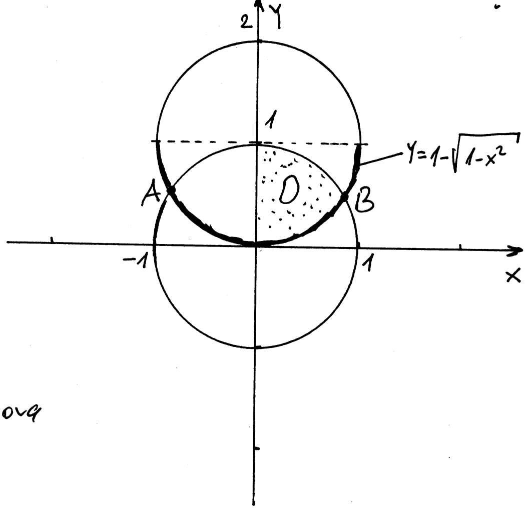
$y=1-\sqrt{1-x^2}$

$y-1=-\sqrt{1-x^2}$

$(y-1)^2=1-x^2$

$x^2+(y-1)^2=1$

krug sa centrom
u $C(0,1)$
poluprečnika $r=1$



Pronađimo tačke presjeka ovih krugova

$$\begin{array}{l} x^2+y^2=1 \\ \underline{x^2+(y-1)^2=1} \\ x^2=1-y^2 \\ \underline{x^2+(y-1)^2=1} \end{array}$$

$1-y^2+(y-1)^2=1$

$1-\cancel{y^2}+\cancel{y^2}-2y+\cancel{1}=\cancel{1}$

$1-2y=0$

$2y=1$

$y=\frac{1}{2}$

$x^2+y^2=1$

$x^2=1-y^2$

$x^2=1-\frac{1}{4}$

$x^2=\frac{3}{4}$

$x_{1,2}=\pm\frac{\sqrt{3}}{2}$

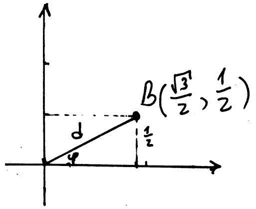
Tačke presjeka su $A(-\frac{\sqrt{3}}{2},\frac{1}{2})$ i $B(\frac{\sqrt{3}}{2},\frac{1}{2})$

Sad možemo konačno nacrtati oblast integracije D.

$$I=\int\limits_{0}^{\sqrt{3}/2} dx \int\limits_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy = \iint\limits_{D} \sqrt{x^2+y^2}\, dx\, dy$$

Oblast D ćemo podjeliti na dva dijela D_1 i D_2 pa ćemo imati

$$\iint\limits_{D} \sqrt{x^2+y^2}\, dx\, dy = \iint\limits_{D_1} \sqrt{x^2+y^2}\, dx\, dy + \iint\limits_{D_2} \sqrt{x^2+y^2}\, dx\, dy.$$

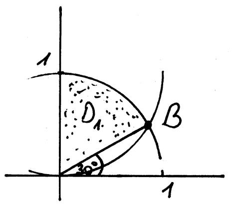


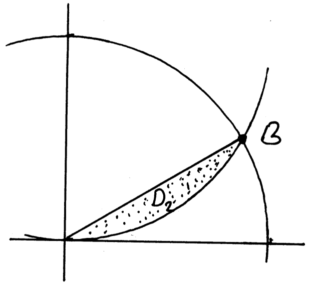
$d=\sqrt{\frac{3}{4}+\frac{1}{4}}=1$

$\sin\varphi=\frac{\frac{1}{2}}{1}$

$\sin\varphi=\frac{1}{2}$

$\varphi=30°$

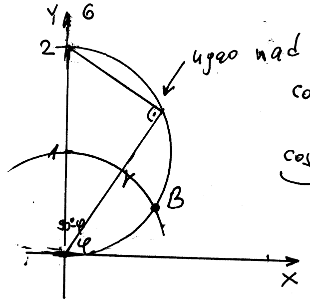




$$\iint\limits_{D_1} \sqrt{x^2+y^2}\, dx\, dy = \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ x=r\cos\varphi \\ y=r\sin\varphi \\ dx\,dy=r\,dr\,d\varphi \end{array} \right.$$

$$D_1 \xrightarrow{\text{transformiše}} D_1' : \begin{cases} 0\le r\le 1 \\ \frac{\pi}{6}\le\varphi\le\frac{\pi}{2} \end{cases} \Bigg| = \iint\limits_{D_1'} \sqrt{r^2}\, r\, dr\, d\varphi = \int\limits_{0}^{1} r^2 dr \int\limits_{\frac{\pi}{6}}^{\pi/2} d\varphi =$$

$$= \varphi\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cdot \frac{1}{3}r^3\Big|_{0}^{1} = \frac{1}{3}\left(\frac{3\pi}{6}-\frac{\pi}{6}\right) = \frac{1}{3}\cdot\frac{2\pi}{6} = \frac{1}{3}\cdot\frac{\pi}{3} = \frac{\pi}{9}$$



$\cos(90°-\varphi)=\frac{r}{2}$

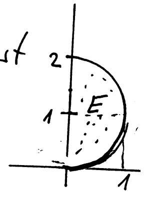
$\underbrace{\cos 90°\cos\varphi+\sin 90°\sin\varphi}_{\cos(90°-\varphi)}=\frac{r}{2}$

$\sin\varphi=\frac{r}{2}$

$r=2\sin\varphi$

Prema tome pomoćna oblast E ima granice

$E: \begin{cases} 0\le r\le 2\sin\varphi \\ 0\le\varphi\le\frac{\pi}{2} \end{cases}$



Odavde možemo vidjeti polarne granice za D_2

$$\iint\limits_{D_2} \sqrt{x^2+y^2}\, dx\, dy = \left| \begin{array}{l} \text{uvedimo polarne koordinate} \\ x=r\cos\varphi \\ y=r\sin\varphi \\ dx\,dy=r\,dr\,d\varphi \end{array} \quad D_2 \xrightarrow{\text{transformiše}} D_2' : \begin{cases} 0\le r\le 2\sin\varphi \\ 0\le\varphi\le\frac{\pi}{6} \end{cases} \right|$$

$$= \iint\limits_{D_2'} r^2\, dr\, d\varphi = \int\limits_{0}^{\frac{\pi}{6}} d\varphi \int\limits_{0}^{2\sin\varphi} r^2 dr = \int\limits_{0}^{\frac{\pi}{6}} \frac{1}{3}r^3\Big|_{0}^{2\sin\varphi} d\varphi = \frac{8}{3}\int\limits_{0}^{\frac{\pi}{6}} \sin^3\varphi\, d\varphi \overset{\text{za vježbu}}{=} \dots = -\sqrt{3}+\frac{16}{9}$$

Prema tome $I=\frac{\pi}{9}+\frac{16}{9}-\sqrt{3}=\frac{\pi+16}{9}-\sqrt{3}$ traženo rješenje

(#) Dati dvostruki integral $\int\limits_{\frac{R}{2}}^{2R} dy \int\limits_{0}^{\sqrt{2Ry-y^2}} f(x,y)\,dx$

iz pravouganih koordinata transformisati na polarne koordinate.

Rj: Skicirajmo oblast integracije.

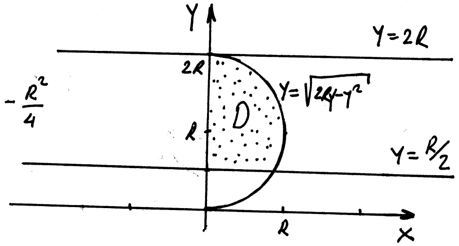
Iz postavke vidimo da je x ograničen sa pravom $x=0$ i krivom $x=\sqrt{2Ry-y^2}$

$$D:\begin{cases} 0\le x\le\sqrt{2Ry-y^2} \\ 2R\le Y\le R/2\end{cases}$$

$$x^2=2Ry-y^2$$
$$x^2+y^2-2\cdot y\cdot R+R^2-R^2=0$$
$$x^2+(Y-R)^2=R^2$$

krug sa centrom u tački $(0,R)$ poluprečnika R.

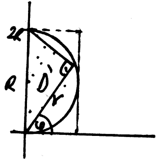
$2R\cdot\frac{R}{2}-\frac{R^2}{4}$



Polarne koordinate glase

$x=r\cos\varphi$
$Y=r\sin\varphi$
$dx\,dy=r\,dr\,d\varphi$

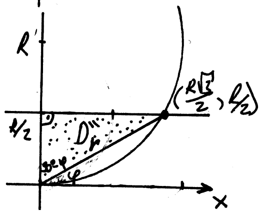
Da bi došli do ideje kako opisati oblast D posmatrajmo sljedeće "jednostavnije" oblasti D' i D'':



$\cos(90°-\varphi)=\frac{r}{2R}$ $\Rightarrow$ $r=2R\sin\varphi$

$\cos(90°-\varphi)=\sin\varphi$

$$D':\begin{cases}0\le r\le 2R\sin\varphi\\ 0\le\varphi\le\frac{\pi}{2}\end{cases}$$



$\cos(90°-\varphi)=\frac{R/2}{r}$

$\sin\varphi=\frac{R}{2r}$

$2r=\frac{R}{\sin\varphi}$ $\Rightarrow$ $r=\frac{R}{2\sin\varphi}$

$$D'':\begin{cases}0\le r\le\frac{R}{2\sin\varphi}\\ \frac{\pi}{6}\le\varphi\le\frac{\pi}{2}\end{cases}$$

Sad nije teško vidjeti da će oblast D opisana pomoću polarnih koordinata postati

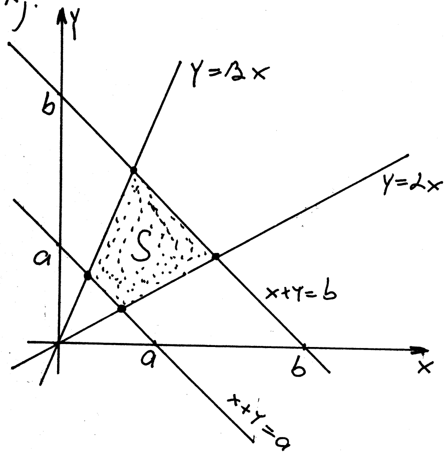
$$D:\begin{cases}\frac{R}{2\sin\varphi}\le r\le 2R\sin\varphi\\ \frac{\pi}{6}\le\varphi\le\frac{\pi}{2}\end{cases}.$$

Prema tome

$$\int\limits_{R/2}^{2R}dy\int\limits_{0}^{\sqrt{2Ry-y^2}}f(x,y)\,dx=\int\limits_{\pi/6}^{\pi/2}d\varphi\int\limits_{\frac{R}{2\sin\varphi}}^{2R\sin\varphi}f(r\cos\varphi,r\sin\varphi)\,r\,dr$$

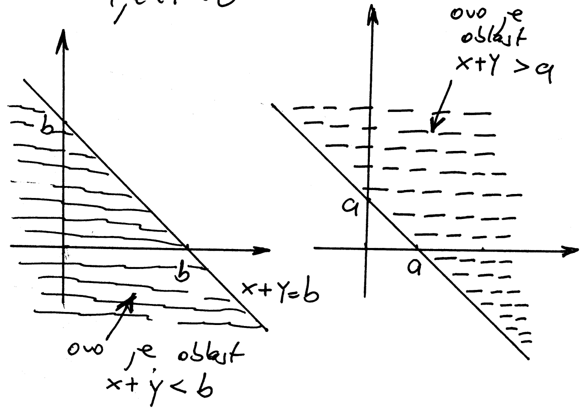
(#) Izračunati integral po oblasti S $\iint_S \frac{1}{xy}\,dx\,dy$ gdje je S oblast ograničena pravama $x+y=a$, $x+y=b$, $y=\alpha x$, $y=\beta x$ gdje su $0<a<b$ i $0<\alpha<\beta$.

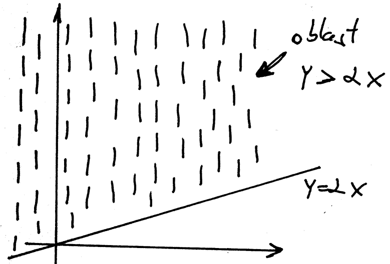
Rj.



Na klasičan način ovaj zadatak nije lagano uraditi. Integral ćemo izračunati uvođenjem smjene.

Primjetimo





Iz ovoga možemo primjetiti da je S oblast gdje je x+y između a i b a $\frac{y}{x}$ između α i β.

$$\iint_S \frac{1}{xy}\,dx\,dy = \left|\begin{array}{ll} u=x+y & a\le u\le b \\ v=\frac{y}{x} & \alpha\le v\le\beta \end{array}\right. \qquad J=\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} \qquad \begin{array}{l} y=vx \\ u=x+vx \\ u=x(1+v) \end{array}$$

$$x=\frac{u}{1+v} \qquad y=\frac{uv}{1+v}$$

$$\frac{\partial x}{\partial u}=\frac{1}{1+v} \qquad \frac{\partial x}{\partial v}=u\cdot(-1)(1+v)^{-2}=\frac{-u}{(1+v)^2} \qquad dx\,dy=|J|\,du\,dv$$

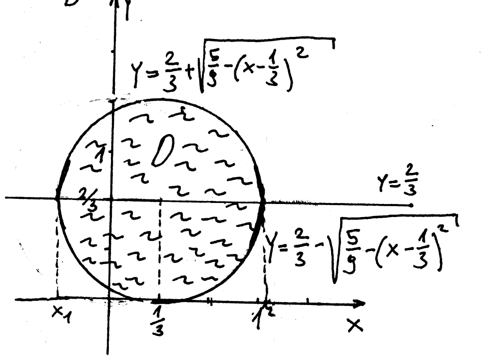
$$\frac{\partial y}{\partial u}=\frac{v}{1+v} \qquad \frac{\partial y}{\partial v}=\frac{u(1+v)-uv\cdot 1}{(1+v)^2}=\frac{u}{(1+v)^2} \qquad J=\frac{u}{(1+v)^3}+\frac{uv}{(1+v)^3}=\frac{u}{(1+v)^2}\ \Bigg| =$$

$$=\int_a^b\left[\int_\alpha^\beta \frac{1}{\frac{u}{1+v}\cdot\frac{uv}{1+v}}\cdot\left|\frac{u}{(1+v)^2}\right|dv\right]du=\int_a^b\left[\int_\alpha^\beta\frac{1}{u}\cdot\frac{1}{v}\,dv\right]du=\int_a^b\frac{1}{u}\ln v\Big|_\alpha^\beta=\ln\frac{\beta}{\alpha}\cdot\ln u\Big|_a^b=\ln\frac{\beta}{\alpha}\ln\frac{b}{a}$$

(#) Izračunati dvostruki integral $I=\iint_D (x^2+y^2)\,dx\,dy$ gdje je $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le\frac{2}{3}(x+2y)\}$.

Rj. Odredimo šta je oblast D.

$$x^2+y^2\le\frac{2}{3}(x+2y)$$
$$x^2+y^2\le\frac{2}{3}x+\frac{4}{3}y$$
$$x^2-\frac{2}{3}x+y^2-\frac{4}{3}y\le 0$$
$$x^2-2\cdot x\cdot\frac{1}{3}+\frac{1}{9}-\frac{1}{9}+y^2-2\cdot y\cdot\frac{2}{3}+\frac{4}{9}-\frac{4}{9}\le 0$$
$$\left(x-\frac{1}{3}\right)^2+\left(y-\frac{2}{3}\right)^2\le\frac{5}{9}$$



D predstavlja unutrašnjost kruga s centrom u tački $\left(\frac{1}{3},\frac{2}{3}\right)$ poluprečnika $r=\frac{\sqrt{5}}{3}\approx 0{,}74$

$$\left(x-\frac{1}{3}\right)^2+\left(y-\frac{2}{3}\right)^2=\frac{5}{9}$$
$$y=\frac{2}{3}$$
$$\left(x-\frac{1}{2}\right)^2=\frac{5}{9}$$
$$x-\frac{1}{3}=\pm\frac{\sqrt{5}}{3}$$
$$x_{1,2}=\frac{1\pm\sqrt{5}}{3}$$

$$\left(y-\frac{2}{3}\right)^2=\frac{5}{9}-\left(x-\frac{1}{3}\right)^2$$
$$y=\frac{2}{3}\pm\sqrt{\frac{5}{9}-\left(x-\frac{1}{3}\right)^2}$$

I način: klasičan način

Nađimo presječnu tačku kruga i prave $y=\frac{2}{3}$

$$I=\iint_D(x^2+y^2)\,dx\,dy=\int_{\frac{1-\sqrt{5}}{3}}^{\frac{1+\sqrt{5}}{3}}\left[\int_{\frac{2}{3}-\sqrt{\frac{5}{9}-(x-\frac{1}{3})^2}}^{\frac{2}{3}+\sqrt{\frac{5}{9}-(x-\frac{1}{3})^2}}(x^2+y^2)\,dy\right]dx=\ldots$$

NA KLASIČAN NAČIN OVO JE TEŠKO URADITI

II način: Uvedimo neku smjenu promjenljivih. Kako je dat krug uvedimo polarne koordinate.

$$x=a+r\cos\varphi \qquad x=\frac{1}{3}+r\cos\varphi$$
$$y=b+r\sin\varphi \quad \text{tj.} \quad y=\frac{2}{3}+r\sin\varphi$$

$0\le\varphi\le 2\pi$

$0\le r\le\frac{\sqrt{5}}{3}$

ove vrijednosti čitamo sa slike

$dx\,dy=r\,dr\,d\varphi$

Jakobijan

$$dx\,dy=|J|\,dr\,d\varphi$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi\end{vmatrix}=r$$

$$x^2+y^2=\left(\frac{1}{3}+r\cos\varphi\right)^2+\left(\frac{2}{3}+r\sin\varphi\right)^2=\underline{\underline{\frac{1}{9}}}+\frac{2}{3}r\cos\varphi+\underline{r^2\cos^2\varphi}+\underline{\underline{\frac{4}{9}}}+\frac{4}{3}r\sin\varphi+\underline{r^2\sin^2\varphi}=$$
$$=\frac{5}{9}+r^2+\frac{2}{3}r\cos\varphi+\frac{4}{3}r\sin\varphi$$

$$I=\iint_D(x^2+y^2)\,dx\,dy=\iint_{D'}\left(\frac{5}{9}+r^2+\frac{2}{3}r(\cos\varphi+2\sin\varphi)\right)r\,dr\,d\varphi=\iint_{D'}\left(\frac{5}{9}r+r^3\right)dr\,d\varphi+$$
$$+\frac{2}{3}\iint_{D'}r^2(\cos\varphi+2\sin\varphi)\,dr\,d\varphi, \quad \iint_{D'}\left(\frac{5}{9}r+r^3\right)dr\,d\varphi=\int_0^{2\pi}\left[\int_0^{\sqrt{5}/3}\left(\frac{5}{9}r+r^3\right)dr\right]d\varphi=2\pi\cdot\left(\frac{5}{9}\cdot\frac{1}{2}r^2\Big|_0^{\frac{\sqrt{5}}{3}}+\right.$$

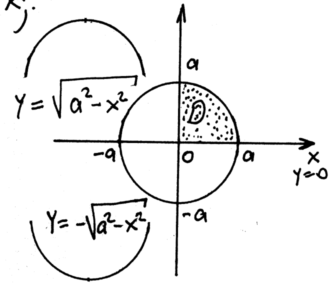
$$+\frac{1}{4}r^4\Big|_0^{\frac{\sqrt5}{3}}\Big) = 2\pi\left(\frac{5}{9\cdot2}\cdot\frac{5}{9}+\frac14\cdot\frac{5\cdot5}{9\cdot9}\right)=\pi\left(\frac{5^2}{9^2}+\frac12\cdot\frac{5^2}{9^2}\right)=\frac32\,\frac{5^2}{9^2}\pi=\frac{25}{54}\pi$$

$$\iint\limits_{D'} r^2(\cos\varphi+2\sin\varphi)\,dr\,d\varphi = \int_0^{\sqrt5/3} r^2\left[\int_0^{2\pi}(\cos\varphi+2\sin\varphi)\,d\varphi\right] = \frac{r^3}{3}\Big|_0^{\frac{\sqrt5}{3}}\left(\sin\varphi\Big|_0^{2\pi}-2\cos\varphi\Big|_0^{2\pi}\right)=0$$

Prema tome $\iint\limits_D (x^2+y^2)\,dx\,dy = \frac{25}{54}\pi$

(#) Izračunati $I=\iint\limits_D\sqrt{x^2+y^2}\,dx\,dy$ gdje je D četvrtina kruga $x^2+y^2\le a^2$.

Rj.



$$\iint\limits_D \sqrt{x^2+y^2}\,dx\,dy = \int_0^a\left(\int_0^{\sqrt{a^2-x^2}}\sqrt{x^2+y^2}\,dy\right)dx =$$

$$= \left| \begin{array}{ll} x=r\cos\varphi & 0\le x\le a\\ y = r\sin\varphi & 0\le y\le\sqrt{a^2-x^2}\\ x^2+y^2=r^2(\cos^2\varphi+\sin^2\varphi)=r^2 & \quad\Downarrow \\ & 0\le\varphi\le\frac{\pi}{2}\\ & 0\le r\le a\end{array}\right.$$

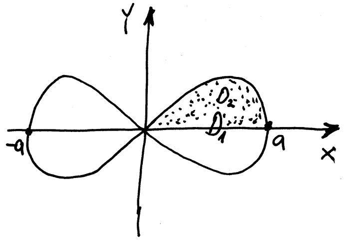
$$dx\,dy = |J|\,dr\,d\varphi$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix} = \begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix} = r \quad\Bigg| \quad = \int_0^a\left[\int_0^{\frac{\pi}{2}}\sqrt{r^2}\,|r|\,d\varphi\right]dr$$

$$= \int_0^a\left[\int_0^{\frac{\pi}{2}} r^2\,d\varphi\right]dr = \int_0^a r^2\cdot\varphi\Big|_0^{\frac{\pi}{2}}\,dr = \int_0^a\frac{\pi}{2}r^2\,dr = \frac{\pi}{2}\cdot\frac13 r^3\Big|_0^a = \frac{a^3\pi}{6}$$

(#) Izračunati dvostruki integral $\iint\limits_D dx\,dy$, ako je D oblast ograničena lemniskatom $(x^2+y^2)^2=a^2(x^2-y^2)$.

Rj. Lemniskata grafički izgleda ovako.



Pronađimo presječne tačke lemniskate sa x-osom: $y=0$

$$x^4=a^2x^2 \Rightarrow x^4-a^2x^2=0$$
$$x^2(x^2-a^2)=0$$
$$x_1=0,\ x_2=a,\ x_3=-a$$

Primjetimo da se površina oblasti D računa po formuli $P=\iint\limits_D dx\,dy$. Naša oblast D je simetrična u odnosu na y-osu pa je $\iint\limits_D dx\,dy = 2\iint\limits_{D_1}dx\,dy$,

Oblast D_1 je simetrična u odnosu na x-osu.

$$\iint\limits_D dx\,dy = 4\iint\limits_{D_2}dx\,dy$$

uvedimo polarne koordinate $x=r\cos\varphi$, $y=r\sin\varphi$, $dx\,dy=r\,dr\,d\varphi$

$$(x^2+y^2)^2=a^2(x^2-y^2)$$
$$(r^2)^2=a^2(r^2\cos^2\varphi-r^2\sin^2\varphi) \quad /:r^2 \ (r\neq0)$$
$$r^2=a^2(\cos^2\varphi-\sin^2\varphi)$$
$$r^2=a^2\cdot\cos2\varphi \Rightarrow r=a\sqrt{\cos2\varphi}$$

(primjetimo da za $\varphi>\frac{\pi}{4}$ r nije definisano)

$$D_2:\begin{cases}0<\varphi<\frac{\pi}{4}\\ 0<r<\sqrt{a^2\cos2\varphi}\end{cases}$$

$$\iint\limits_D dx\,dy = 4\iint\limits_{D_2}r\,dr\,d\varphi = 4\int_0^{\frac{\pi}{2}}d\varphi\int_0^{a\sqrt{\cos2\varphi}}r\,dr = 4\int_0^{\frac{\pi}{4}}\frac12 r^2\Big|_0^{a\sqrt{\cos2\varphi}}d\varphi = \frac{4}{2}\int_0^{\frac{\pi}{4}}a^2\cos2\varphi\,d\varphi =$$

$$= 2a^2\cdot\frac12\sin2\varphi\Big|_0^{\frac{\pi}{4}} = a^2\left(\sin\frac{\pi}{2}-0\right)=a^2 \quad \text{traženo rješenje}$$

Oblast D preslikati pomoću preslikavanja f: $(x, y)\to(u\ v)$.

108. D je oblast ograničena parabolama $y^2=px$, $y^2=qx$, $x^2=ay$, $x^2=by$, $0<p<q$, $0<a<b$, a preslikavanje f je dato jednakostima $y^2=ux$, $x^2=vy$.

Rješenja:

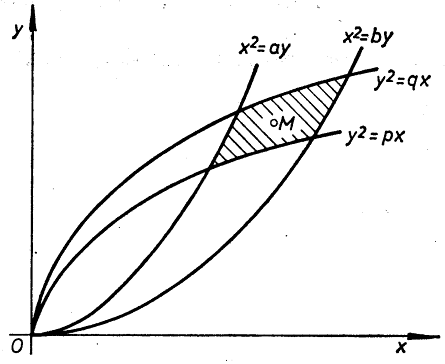
108. Odredićemo Jakobijan preslikavanja. Kako je $u=\frac{y^2}{x}$, $v=\frac{x^2}{y}$, to je:

$$\frac{D(u, v)}{D(x, y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}-\frac{y^2}{x^2} & \frac{2y}{x}\\ \frac{2x}{y} & -\frac{x^2}{y^2}\end{vmatrix}=1-\frac{4xy}{xy}=1-4=-3.$$

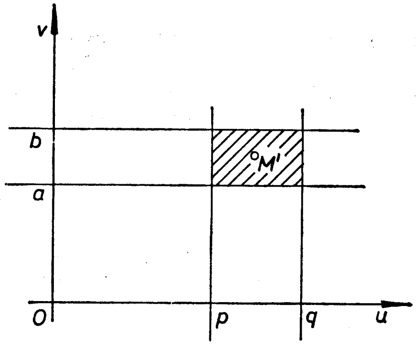
Dakle,

$$J=\frac{D(x, y)}{D(u, v)}=-\frac{1}{3}\neq 0,$$

pa je preslikavanje obostrano jednoznačno. Slike datih parabola su prave $u=p$, $u=q$, $v=a$, $v=b$, a oblast D (sl. 24) se preslikava na pravougaonik D' (sl. 25). Tačka $M\in D$ preslikava se na $M'\in D'$.



Sl. 24



Sl. 25

Oblast D preslikati pomoću preslikavanja f: $(x, y)\to(u\ v)$.

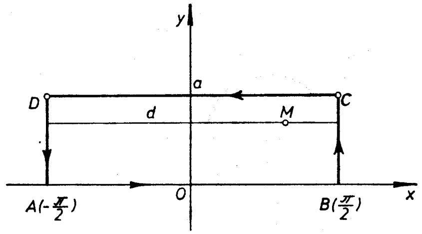
109. $D=\left\{(x, y):-\frac{\pi}{2}\leqslant x\leqslant\frac{\pi}{2},\ 0\leqslant y\leqslant a\right\}$, a preslikavanje f je dato sa $u=\sin x\operatorname{ch} y$, $v=\cos x\operatorname{sh} y$.

Rješenja:

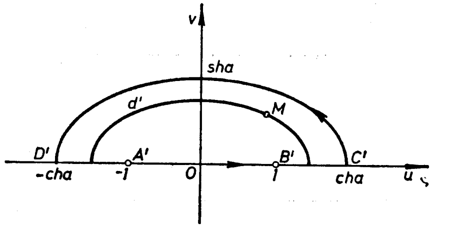
109. Preslikavanje je obostrano jednoznačno na $D\setminus\{A, B\}$, jer je

$$\frac{D(u, v)}{D(x, y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\cos x\operatorname{ch} y & \sin x\operatorname{sh} y\\ -\sin x\operatorname{sh} y & \cos x\operatorname{ch} y\end{vmatrix}=\cos^2 x\operatorname{ch}^2 y+\sin^2 x\operatorname{sh}^2 y\neq 0$$

za $(x, y)\in D\setminus\{A, B\}$. Dio $-\frac{\pi}{2}\leqslant x\leqslant\frac{\pi}{2}$, $y=0$ preslikava se na dio $-1\leqslant u\leqslant 1$, $v=0$ prave $v=0$ (sl. 26 i 27). Duž BC ima jednačinu: $x=\frac{\pi}{2}$, $0\leqslant y\leqslant a$, pa je njena slika skup tačaka (u, v) za koje je $u=\operatorname{ch} y$, $v=0$.



Sl. 26



Sl. 27

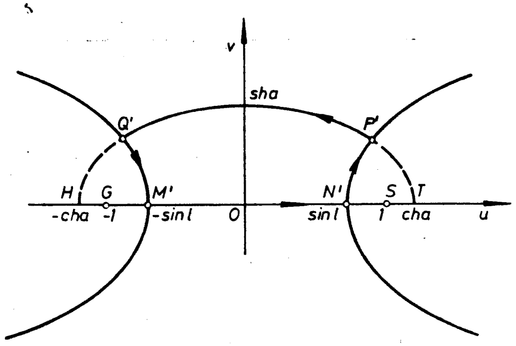
Duž DC ima jednačinu $y=a$, $-\frac{\pi}{2}\leqslant x\leqslant\frac{\pi}{2}$, pa se preslikava na skup tačaka (u, v) za koje je $u=\sin x\operatorname{ch} a$, $v=\cos x\operatorname{sh} a$, $v\geqslant 0$, tj. na gornju polovinu elipse

$$\frac{u^2}{\operatorname{ch}^2 a}+\frac{v^2}{\operatorname{sh}^2 a}=1.$$

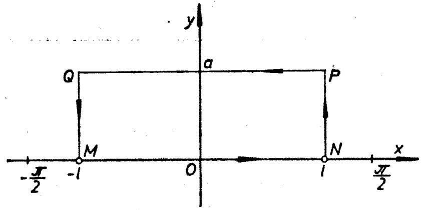
Duž DA preslikava se na duž $D'A'$ (sl. 26 i 27).

Da se unutrašnja tačka M oblasti D preslikava u unutrašnju tačku oblasti gornje poluelipse, može se zaključiti na sljedeći način. Kroz tačku M uočimo duž d paralelnu sa duži AB; njena slika će biti gornji luk elipse čije su poluose manje od ch a i sh a, pa kako $M\in d \Rightarrow M'\in d'$, to slijedi zaključak.

Primjedba. Neka student sam nađe sliku pravougaonika $D=\{(x, y): -l\leqslant x\leqslant l,\ 0\leqslant y\leqslant a,\ 0\leqslant l\leqslant \frac{\pi}{2}$ (sl. 28 a i b). (Prava $x=l$ se preslikava na skup tačaka (u, v) za koje je $u=\sin l\,\mathrm{ch}\,y$, $v=\cos l\,\mathrm{sh}\,y$, tj. na skup tačaka (u, v) hiperbole $\frac{u^2}{\sin^2 l}-\frac{v^2}{\cos^2 l}=\mathrm{ch}^2\,y-\mathrm{sh}^2\,y=1$.)



Sl. 28 a



Sl. 28 b

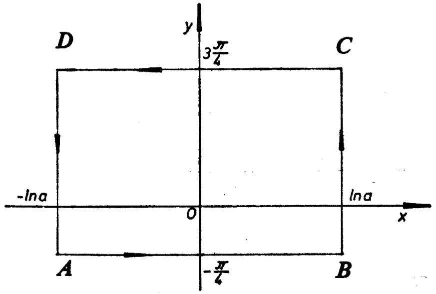
Kada $l\to\frac{\pi}{2}$, onda figura $M'N'P'Q'$ (sl. 28 a) postaje gornja poluelipsa, tj. $N'P'$ (luk hiperbole) teži duži ST.

(#) Oblast D preslikati pomoću preslikavanja f: $(x, y)\to(u\ v)$.

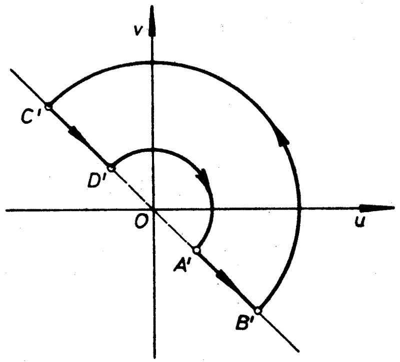
110. $D=\left\{(x, y): |x|\leqslant \ln a,\ -\frac{1}{4}\pi\leqslant y\leqslant\frac{3}{4}\pi\right\}$, a preslikavanje f je dato sa $u=e^x\cos y$, $v=e^x\sin y$.

Rješenja:

110. Odredimo sliku konture oblasti $\mathcal{D}$ (sl. 29 a).



Sl. 29 a



Sl. 29 b

Duž AB ima jednačinu $y=-\frac{\pi}{4}$, $-\ln a\leqslant x\leqslant\ln a$, pa će biti (sl. 29 b).

$$A'B'=\left\{(u, v): u=\frac{e^x}{\sqrt{2}},\ v=-\frac{e^x}{\sqrt{2}},\ -\ln a\leqslant x\leqslant\ln a\right\},$$

dakle, $A'B'$ je dio prave $v=-u$, pri čemu je $v<0$, i to $-\frac{a}{\sqrt{2}}\leqslant v\leqslant-\frac{a^{-1}}{\sqrt{2}}$.

Na isti način zaključujemo da duž CD ima sliku $C'D'$, duž na pravoj $v=-u$,

$$\frac{a^{-1}}{\sqrt{2}}\leqslant v\leqslant\frac{a}{\sqrt{2}}.$$

Duž BC ima jednačinu $x=\ln a$, $-\frac{\pi}{4}\leqslant y\leqslant\frac{3\pi}{4}$, pa će njena slika biti skup tačaka $\{(u, v): u=a\cdot\cos y,\ v=a\ \sin y\}$. Dakle, to je dio kružnice poluprečnika a.

Na isti način se zaključuje da duž DA ima kao sliku dio kružnice poluprečnika a^{-1}.

Preslikavanje je obostrano jednoznačno jer je

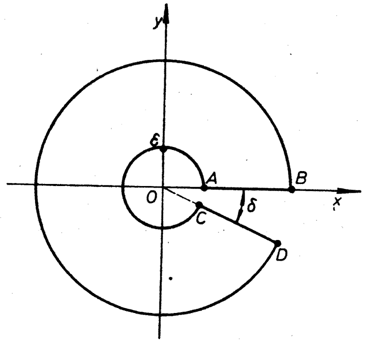
$$\frac{D(u, v)}{D(x, y)}=e^x>0, \quad \text{za svako } x.$$

(#) Oblast D preslikati pomoću preslikavanja $f\colon (x, y) \to (u\ v)$.

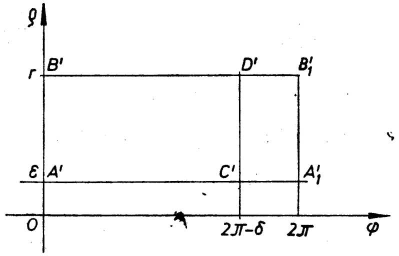
111. $D = \{(x, y) : x^2 + y^2 \leqslant r^2\}$, a preslikavanje f je dato sa $x = \rho\cos\varphi$, $y = \rho\sin\varphi$.

Rješenja:

111. Kako je $\dfrac{D(x, y)}{D(\rho, \varphi)} = \rho$, a u tački $(0, 0) \in D$ je $\rho = 0$, to ćemo najprije naći sliku oblasti $G \subset D$ koja je određena dijelovima kružnica poluprečnika r i ε, dužima AB i CD, pri čemu duž AB leži na x-osi, a duž CD na polupravoj čija je početna tačka $O(0, 0)$ i koja gradi ugao $2\pi - \delta$ (odnosno δ) sa polupravom OB (sl. 30a). Oblast G se preslikava na pravougaonik $A'B'D'C'$ (sl. 31a).



Sl. 30a

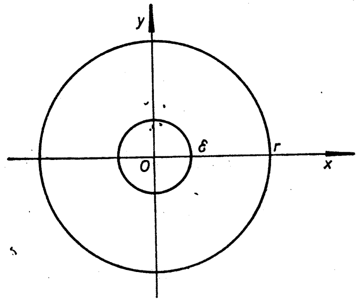


Sl. 31a

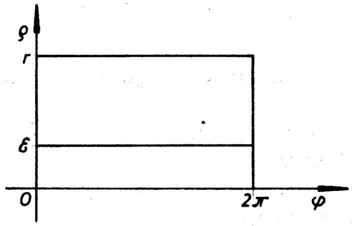
Ako pustimo da $\delta \to 0$, onda tačka $C \to A$, $D \to B$, i $D' \to B_1'$, $C' \to A_1'$.

Dakle, duži AB u ovom preslikavanju odgovaraju i duž $A'B'$ i duž $A_1'B_1'$.

Kružni prsten određen kružnicama poluprečnika r i ε preslikava se na pravougaonik određen pravama $\rho = \varepsilon$, $\rho = r$, $\varphi = 0$, $\varphi = 2\pi$ (sl. 30b, 31b).



Sl. 30b



Sl. 31b

Ako sada pustimo da $\varepsilon \to 0$, onda slika ε kružnice (duž) teži duži $[0, 2\pi]$ na pravoj $\rho = 0$ u sistemu $O\rho\varphi$. To znači da u ovom preslikavanju tački $(0, 0)$ odgovara duž $[0, 2\pi]$. Krug poluprečnika r preslikava se na pravougaonik $\rho = 0$, $\rho = r$, $\varphi = 0$, $\varphi = 2\pi$.

Primjedba. Neka student uoči značenja veličina ρ i φ u koordinatnom sistemu Oxy.

(#) Pomoću smjene promjenljivih izračunati integrale:

114. $\displaystyle\iint_D \sqrt{r^2 - (x^2 + y^2)}\, dxdy$, gdje je D oblast ograničena kružnicom $x^2 + y^2 - rx = 0$.

115. $\displaystyle\iint_D \ln(x^2 + y^2)\, dxdy$, gdje je D oblast ograničena kružnicama $x^2 + y^2 = e^2$ i $x^2 + y^2 = e^4$.

Rješenja:

114. Uvodeći smjenu promjenljivih $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, podintegralna funkcija postaje $\sqrt{a^2 - \rho^2}$, pa kako je $|J| = \rho$, biće

$$I = \iint_{D'} \sqrt{a^2 - \rho^2}\ \rho\, d\rho\, d\varphi.$$

Jednačina kružnice u novim koordinatama je:

$$x^2 + y^2 - rx = \rho^2\cos^2\varphi + \rho^2\sin^2\varphi - r\rho\cos\varphi = 0,$$

tj. $\rho = r\cos\varphi$.

Otuda je

$$I = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{\rho(\varphi)} \sqrt{r^2 - \rho^2}.\ \rho\, d\rho = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{r\cos\varphi} \rho\sqrt{r^2 - \rho^2}\, d\rho =$$

$$= -\frac{1}{3}\int_{-\pi/2}^{\pi/2} (r^2 - \rho^2)^{3/2} \Big|_0^{r\cos\varphi} d\varphi = \frac{r^3}{3}\int_{+\pi/2}^{\pi/2} (1 - \sin^3\varphi)\, d\varphi = \frac{r^3\pi}{3}.$$

115. Smjenom $x = \rho\cos\varphi$, $y = \rho\sin\varphi$ dobija se

$$\iint_D \ln(x^2 + y^2)\, dxdy = 2\iint_{D'} \rho\ln\rho\ d\rho\ d\varphi = 2\int_0^{2\pi} d\varphi \int_e^{e^2} \rho\ln\rho\ d\rho =$$

$$= 4\pi\int_e^{e^2} \rho\ln\rho\ d\rho = 4\pi\left[\frac{1}{2}\rho^2\ln\rho - \frac{1}{4}\rho^2\right]_e^{e^2} = \pi e^2(3e^2 - 1).$$

(Za izračunavanje integrala $\int \rho\ln\rho\, d\rho$ primijenjena je parcijalna integracija.)

Pomoću smjene promjenljivih izračunati integrale:

116. $I(r)=\iint\limits_{D} e^{-x^2-y^2}\,dxdy$, gdje je D oblast ograničena kružnicom $x^2+y^2=r^2$. Naći lim $I(r)$ kad $r\to\infty$.

117. $\displaystyle\iint\limits_{D}\frac{dxdy}{(x^2+y^2)\left(1+\sqrt[3]{x^2+y^2}\right)}$. $D=\{(x,\ y): x^2-y^2\leqslant 0,\ 1\leqslant x^2+$ $+y^2\leqslant 4\}$.

Rješenja: **116.** $I(r)=\int\limits_{0}^{2\pi} d\varphi \int\limits_{0}^{r} e^{-\rho^2}\rho\ d\rho=(1-e^{-r^2})\,\pi$,

$\lim\limits_{r\to\infty} I(r)=\pi$. Ovo znači da je

$$\Big(\int\limits_{-\infty}^{\infty} e^{-t^2}\,dt\Big)^2=\int\limits_{-\infty}^{\infty} e^{-x^2}dx\int\limits_{-\infty}^{\infty} e^{-y^2}dy=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-x^2-y^2}\,dxdy=\pi,$$

tj.

$$\int\limits_{-\infty}^{\infty} e^{-t^2}\,dt=\sqrt{\pi}.$$

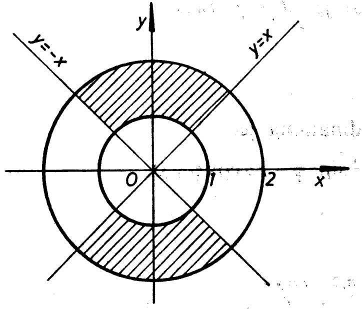
117. Najprije skiciraćemo oblast integracije. Biće:

$\{(x,\ y): x^2-y^2\leqslant 0\}=\{(x,\ y):(x-y)(x+y)\leqslant 0\}=\{(x,\ y): x<y\wedge x>-y$ ili $x>y\wedge x<-y\}$,

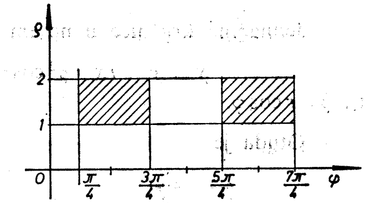
odnosno

$$\{(x,\ y): x^2-y^2\leqslant 0\}=\{(x,\ y): x^2\leqslant y^2\}=\{(x,\ y): |x|\leqslant|y|\}.$$

Oblast integracije D prikazana je na sl. 32a.



Sl. 32a



Sl. 32b

Smjenom $x=\rho\cos\varphi$, $y=\rho\sin\varphi$, data oblast D preslikava se na oblast D' (sl. 32a i b), pa je:

$$I=\iint\limits_{D'}\frac{d\rho\,d\varphi}{\rho\left(1+\sqrt[3]{\rho^2}\right)}=\int\limits_{\pi/4}^{3\pi/4} d\varphi\int\limits_{1}^{2}\frac{d\rho}{\rho\left(1+\sqrt[3]{\rho^2}\right)}+\int\limits_{5\pi/4}^{7\pi/4} d\varphi\int\limits_{1}^{2}\frac{d\rho}{\rho\left(1+\sqrt[3]{\rho^2}\right)}=$$

$$=\pi\int\limits_{1}^{2}\frac{d\rho}{\rho\left(1+\sqrt[3]{\rho^2}\right)}.$$

Smjenom $\sqrt[3]{\rho^2}=t$ dobija se

$$I=\frac{3\pi}{2}\int\limits_{1}^{\sqrt[3]{4}}\frac{1}{t(t+1)}\,dt=\frac{3\pi}{2}\ln\frac{t}{t+1}\Bigg|_{1}^{\sqrt[3]{4}}=\frac{\pi}{2}\ln\frac{32}{\left(1+\sqrt[3]{4}\right)^3}.$$

Pomoću smjene promjenljivih izračunati integral:

120. $\iint\limits_{D}(x+y)^p(x-y)^q\,dxdy$, D je oblast ograničena pravama $x+y=1$, $x-y=1$, $x+y=3$, $x-y=-1$, p realan a q prirodan broj.

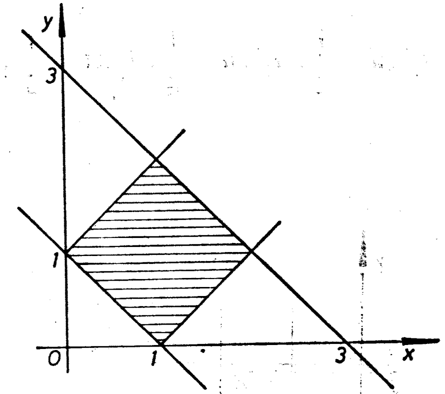
Rješenja:

120. Koristićemo smjenu

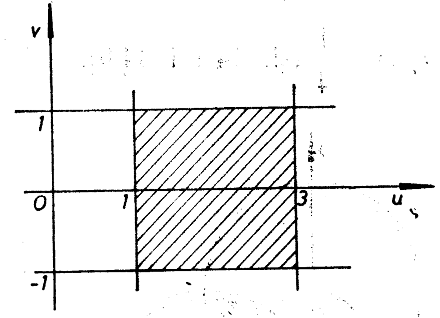
$$x+y=u,\quad x-y=v \Leftrightarrow x=\frac{1}{2}(u+v),\quad y=\frac{1}{2}(u-v).$$

Oblast D (kvadrat na sl. 33a) preslikava se na kvadrat D' (sl. 33b.); preslikavanje je obostrano jednoznačno, jer je

$$J=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}\dfrac{1}{2} & \dfrac{1}{2}\\[2ex] \dfrac{1}{2} & -\dfrac{1}{2}\end{vmatrix}=-\frac{1}{2}\,(\neq 0).$$



Sl. 33a



Sl. 33b

Biće: $I=\displaystyle\iint\limits_{D'} u^p v^q\,|J|\,dudv=\frac{1}{2}\int\limits_{1}^{3} u^p\,du\int\limits_{-1}^{1} v^q\,dv=\frac{1}{2}\,\frac{u^{p+1}}{p+1}\Bigg|_{1}^{3}\cdot\frac{v^{q+1}}{q+1}\Bigg|_{-1}^{1}=$

$$=\frac{1}{2(p+1)(q+1)}\cdot(3^{p+1}-1)\cdot[1-(-1)^{q+1}]\ \text{za}\ p\neq-1,\quad q\neq-1.$$

Konačno, $I=0$ za $q=2k-1$, $\pm k=1,\ 2,\ \ldots$; $I=\dfrac{3^{p+1}-1}{(p+1)(q+1)}$ za $q=2k$, $\pm k=0,\ 1,\ 2,\ \ldots$

Neka student samostalno riješi slučaj $p=-1\vee q=-1$.

(#) Pomoću smjene promjenljivih izračunati integral:

122. $\iint\limits_D (x^2+y^2)^{-2}\,dxdy$, gdje je D oblast ograničena kružnicama $l_1: x^2+y^2-2x=0$, $l_2: x^2+y^2-4x=0$; $l_3: x^2+y^2-2y=0$, $l_4: x^2+y^2-4y=0$.

Rješenja:

122. Napisaćemo jednačine kružnica u obliku

$$l_1: 1-2\frac{x}{x^2+y^2}=0,\quad l_2: 1-4\frac{x}{x^2+y^2}=0,$$

$$l_3: 1-2\frac{y}{x^2+y^2}=0,\quad l_4: 1-4\frac{y}{x^2+y^2}=0$$

i koristićemo smjenu

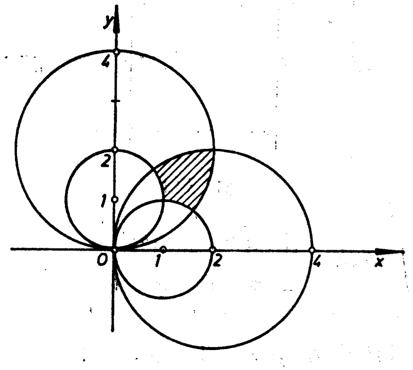
$$\frac{x}{x^2+y^2}=u,\ \frac{y}{x^2+y^2}=v \Leftrightarrow \frac{u}{u^2+v^2}=x,\ \frac{v}{u^2+v^2}=y.$$

Pri tome je

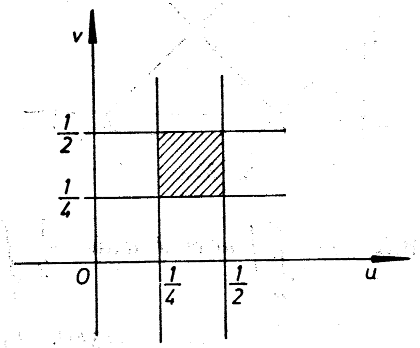
$$\frac{D(x,y)}{D(u,v)}=-\frac{1}{(u^2+v^2)^2},\quad u^2+v^2=\frac{1}{x^2+y^2}.$$

Sada je $\qquad I=\iint\limits_{D'} dudv,$

pri čemu je oblast D' ograničena pravama $l_1': u=\frac{1}{2}$, $l_2': u=\frac{1}{4}$, $l_3': v=\frac{1}{2}$, $l_4': v=\frac{1}{4}$ (sl. 34 a i 34 b).



Sl. 34 a

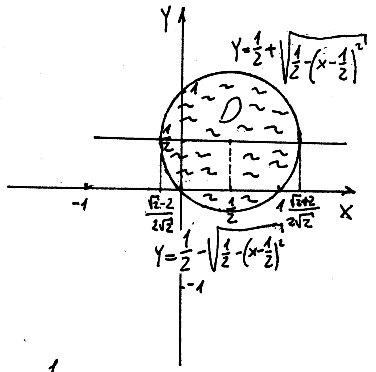


Sl. 34 b

Biće

$$I=\frac{1}{4}\cdot\frac{1}{4}=\frac{1}{16}.$$

(#) Izračunati dvostruki integral: $I=\iint\limits_D (x+y)\,dxdy$, gdje je $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le x+y\}$.



Rj.
$$x^2+y^2\le x+y$$
$$x^2-x+y^2-y\le 0$$
$$x^2-2\cdot x\cdot\frac{1}{2}+\frac{1}{4}-\frac{1}{4}+y^2-2\cdot y\cdot\frac{1}{2}+\frac{1}{4}-\frac{1}{4}\le 0$$
$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2\le\frac{1}{2}$$

Unutrašnjost
Vkruga sa centrom u tački $S(\frac{1}{2},\frac{1}{2})$
poluprečnika $r=\frac{1}{\sqrt{2}}\approx 0{,}7$.

Naćimo presječne tačke kruga sa pravom $y=\frac{1}{2}$

$$\frac{\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}}{y=\frac{1}{2}}\qquad \left(x-\frac{1}{2}\right)^2=\frac{1}{2}\qquad x_1=\frac{1}{2}+\frac{1}{\sqrt{2}}=\frac{2+\sqrt{2}}{2\sqrt{2}}$$

$$x-\frac{1}{2}=\pm\frac{1}{\sqrt{2}}\qquad x_2=\frac{1}{2}-\frac{1}{\sqrt{2}}=\frac{\sqrt{2}-2}{2\sqrt{2}}$$

I način:

$$I=\iint\limits_D (x+y)\,dxdy=\int_{\frac{\sqrt{2}-2}{2\sqrt{2}}}^{\frac{\sqrt{2}+2}{2\sqrt{2}}}\left[\int_{\frac{1}{2}-\sqrt{\frac{1}{2}-(x-\frac{1}{2})^2}}^{\frac{1}{2}+\sqrt{\frac{1}{2}-(x-\frac{1}{2})^2}}(x+y)\,dy\right]dx=\dots$$

KOMPLIKOVANO

$$\boxed{\begin{aligned}\left(y-\frac{1}{2}\right)^2&=\frac{1}{2}-\left(x-\frac{1}{2}\right)^2\\ y-\frac{1}{2}&=\pm\sqrt{\frac{1}{2}-\left(x-\frac{1}{2}\right)^2}\\ y&=\frac{1}{2}\pm\sqrt{\frac{1}{2}-\left(x-\frac{1}{2}\right)^2}\end{aligned}}$$

II način: Uvedimo neku smjenu promjenjivih.
Kako je u pitanju krug, uvedimo polarne koordinate.

$$x=a+r\cos\varphi \qquad \text{tj.}\qquad x=\frac{1}{2}+r\cos\varphi \qquad 0\le\varphi\le 2\pi$$
$$y=b+r\sin\varphi \qquad\qquad y=\frac{1}{2}+r\sin\varphi \qquad 0\le r\le\frac{\sqrt{2}}{2}$$

→ ove vrijednosti čitamo sa slike

Jakobijan
↓
$$dxdy=|J|\,drd\varphi$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}=\begin{vmatrix}\cos\varphi & -r\sin\varphi\\ \sin\varphi & r\cos\varphi\end{vmatrix}=r\cos^2\varphi+r\sin^2\varphi=r.$$

$$dxdy=r\,drd\varphi$$

$$I=\iint\limits_D (x+y)\,dxdy=\iint\limits_{D'}\left(\frac{1}{2}+r\cos\varphi+\frac{1}{2}+r\sin\varphi\right)r\,drd\varphi=\iint\limits_{D'}\left(r+r^2(\cos\varphi+\sin\varphi)\right)drd\varphi$$

$$=\iint\limits_{D'} r\,drd\varphi+\iint\limits_{D'} r^2(\cos\varphi+\sin\varphi)\,drd\varphi,\quad \iint\limits_{D'} r\,drd\varphi=\int_0^{2\pi}\left[\int_0^{\frac{\sqrt{2}}{2}} r\,dr\right]d\varphi=2\pi\cdot\frac{1}{2}r^2\Big|_0^{\frac{\sqrt{2}}{2}}=\frac{\pi}{2}$$

$$\iint_{0} r^2(\cos\varphi+\sin\varphi)\,dr\,d\varphi=\int_0^{2/\sqrt{2}} r^2\left[\int_0^{2\pi}(\cos\varphi+\sin\varphi)\,d\varphi\right]dr=\frac{1}{3}r^3\Big|_0^{\frac{2}{\sqrt{2}}}\cdot\left(\sin\varphi\Big|_0^{2\pi}-\cos\varphi\Big|_0^{2\pi}\right)$$

$$=\frac{1}{3}\cdot\frac{8}{2\sqrt{2}}\left(0-(1-1)\right)=0$$

Prema tome:

$$\iint_{0}(x+y)\,dx\,dy=\frac{\pi}{2}$$

Zadaci za vježbu

U zadacima 3525 — 3531 transformisati dvojni integral $\iint_D f(x,\ y)\,dx\,dy$ na polarne koordinate ρ i φ ($x=\rho\cos\varphi$. $y=\rho\sin\varphi$), a zatim ga svesti na dvostvruki (sa određenim posebnim granicama integracije).

3525. D je krug: 1) $x^2+y^2<R^2$; 2) $x^2+y^2<ax$; 3) $x^2+y^2<by$.

3526. D je oblast ograničena kružnim linijama $x^2+y^2=4x$, $x^2+y^2=8x$ i pravama $y=x$ i $y=2x$.

3527. D je oblast koja predstavlja zajednički deo dva kruga $x^2+y^2<ax$ i $x^2+y^2<by$.

3528. D je oblast ograničena pravama $y=x$, $y=0$ i $x=1$.

3529. D je odsečak koji prava $x+y=2$ odseca od kruga $x^2+y^2=4$.

3530. D je oblast ograničena desnom petljom lemniskate $(x^2+y^2)^2=$ $=a^2(x^2-y^2)$.

3531. D je oblast određena nejednakostima $x>0$, $y\geqslant 0$, $(x^2+y^2)^3<$ $<4a^2x^2y^2$.

U zadacima 3532 — 3535 date dvostruke integrale transformisati na polarne koordinate.

3532. $\int_0^R dx\int_0^{\sqrt{R^2-x^2}} f(x,\ y)\,dy$. **3533.** $\int_{\frac{R}{2}}^{2R} dy\int_0^{\sqrt{2Ry-y^2}} f(x,\ y)\,dx$.

3534. $\int_0^R dx\int_0^{\sqrt{R^2-x^2}} f(x^2+y^2)\,dy$.

3535. $\int_0^{\frac{R}{\sqrt{1+R^2}}} dx\int_0^{Rx} f\left(\frac{y}{x}\right)dy+\int_{\frac{R}{\sqrt{1+R^2}}}^{R} dx\int_0^{\sqrt{R^2-x^2}} f\left(\frac{y}{x}\right)dy$.

U zadacima 3536 — 3540 izračunati date dvojne integrale prelazeći na polarne koordinate.

3536. $\iint_D \ln(1+x^2+y^2)\,dx\,dy$, oblast D je četvrtina kruga $x^2+y^2<R^2$ koja leži u prvom kvadrantu.

3537. $\iint_D\sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,dx\,dy$, oblast D je određena nejednakostima $x^2+y^2<1$, $x>0$, $y>0$.

3538. $\iint_D (h-2x-3y)\,dx\,dy$, D je krug $x^2+y^2<R^2$.

3539. $\iint_D \sqrt{R^2-x^2\quad y^2}\,dx\,dy$, D je krug $x^2+y^2<Rx$.

3540. $\iint_D \operatorname{arctg}\frac{y}{x}\,dx\,dy$, D je deo prstena $x^2+y^2>1$, $x^2+y^2<9$, $y\geqslant\frac{x}{\sqrt{3}}$, $y<x\sqrt{3}$.

Rješenja

3525. 1) $\int_0^{2\pi} d\varphi\int_0^{0} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$;

2) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi\int_0^{\alpha\cos\varphi} f(\rho\cos\varphi,\ \rho\sin\rho)\,\rho\,d\rho$;

3) $\int_0^{\pi} d\varphi\int_0^{b\sin\varphi} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3526. $\int_{\frac{\pi}{4}}^{\operatorname{arctg}2} d\varphi\int_{4\cos\varphi}^{8\cos\varphi} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3527. $\int_0^{\operatorname{arctg}\frac{a}{b}} d\varphi\int_0^{b\sin\varphi} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho+$

$\int_{\operatorname{arctg}\frac{a}{b}}^{\frac{\pi}{2}} d\varphi\int_0^{a\cos\varphi} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3528. $\int_0^{\frac{\pi}{4}} d\varphi\int_0^{\sec\varphi} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3529. $\int_0^{\frac{\pi}{2}} d\varphi\int_{\sqrt{2}\sec\left(\varphi-\frac{\pi}{4}\right)}^{2} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3530. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi\int_0^{a\sqrt{\cos2\varphi}} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3531. $\int_0^{\frac{\pi}{2}} d\varphi\int_0^{a\sin2\varphi} f(\rho\cos\varphi),\ \rho\sin\varphi)\,\rho\,d\rho$.

3532. $\int_0^{\frac{\pi}{2}} d\varphi\int_0^{R} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3533. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\varphi\int_{\frac{R}{2\sin\varphi}}^{2R\sin\varphi} f(\rho\cos\varphi,\ \rho\sin\varphi)\,\rho\,d\rho$.

3541. Na osnovu geometrijskih razmatranja pokazati da: ako se dekartove kordinate transformišu shodno obrascima $x=a\rho\cos\varphi$, $y=b\rho\sin\varphi$, u kojima su a i b konstante, onda će elemenat površine biti $d\sigma=ab\rho\,d\rho\,d\varphi$.

U zadacima 3542 — 3544 koristeći rezultat prethodnog zadatka i izabravši najpogodnije vrednosti za a i b, transformisati dvojne integrale.

3542. $\iint_D f(x,\ y)\,dx\,dy$. D je oblast ograničena elipsom $\frac{x^2}{4}+\frac{y^2}{9}=1$.

3543. $\iint_D f(x,\ y)\,dx\,dy$. D je oblast ograničena krivom $\left(x^2+\frac{y^2}{3}\right)^2=x^2y$.

3544. $\iint_D f\left(\sqrt{4-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\right)dx\,dy$, D je deo eliptičnog prstena ograničenog elipsama $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ i $\frac{x^2}{4a^2}+\frac{y^2}{4b^2}=1$, koji leži u prvom kvadrantu.

3545. Izračunati integral $\iint_D xy\,dx\,dy$, u kojem je D oblast u prvom kvadrantu, ograničena elipsom $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

3546. Izračunati integral $\iint_D \sqrt{xy}\,dx\,dy$, u kojem je D oblast u prvom kvadrantu, ograničena krivom $\left(\frac{x^2}{2}+\frac{y^2}{b}\right)^4=\frac{xy}{\sqrt{6}}$.

Rješenja

3534. $\frac{\pi}{2}\int_0^R f(\rho^2)\,\rho\,\rho d$. **3535.** $\frac{R^2}{2}\int_0^{\operatorname{arctg}R} f(\operatorname{tg}\varphi)\,d\varphi$.

3536. $\frac{\pi}{4}[(1+R^2)\ln(1+R^2)-R^2]$. **3537.** $\frac{\pi(\pi-2)}{8}$. **3538.** πR^2h.

3539. $\frac{R^3}{3}\left(\pi-\frac{4}{3}\right)$. **3540.** $\frac{\pi^2}{6}$.

3542. $x=2\rho\cos\varphi$, $y=3\rho\sin\varphi$; $I=6\int_0^{2\pi} d\varphi\int_0^{1} f(2\rho\cos\varphi,\ 3\rho\sin\varphi)\,\rho\,d\rho$.

3543. $x=\rho\cos\varphi$, $y=\sqrt{3}\,\rho\sin\varphi$;

$$I=\sqrt{3}\int_0^{\pi} d\varphi\int_0^{\sqrt{3}\cos^2\varphi\sin\varphi} f(\rho\cos\varphi,\ \sqrt{3}\,\rho\sin\varphi)\,\rho\,d\rho.$$

3544. $x=a\rho\cos\varphi$, $y=b\rho\sin\varphi$; $I=ab\int_0^{\frac{\pi}{2}} d\varphi\int_1^{2} f(\sqrt{4-\rho^2})\,\rho\,d\rho$.

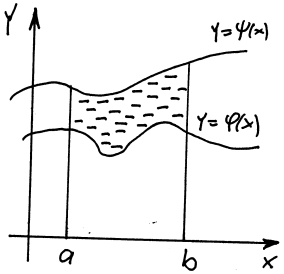
3545. $\frac{a^2b^2}{8}$. **3546.** $\frac{1}{\overset{4}{\sqrt{6}}}$.

Trostruki integral

$$I=\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz\,,$$ Ω oblast integracije u prostoru

ako je $\Omega:\begin{cases} a\le x\le b\\ \varphi(x)\le y\le\psi(x)\\ \alpha(x,y)\le z\le\beta(x,y)\end{cases}$ tada

$$I=\int_a^b dx\int_{\varphi(x)}^{\psi(x)}dy\int_{\alpha(x,y)}^{\beta(x,y)} f(x,y,z)\,dz$$



Oblast Ω možemo projicirati na

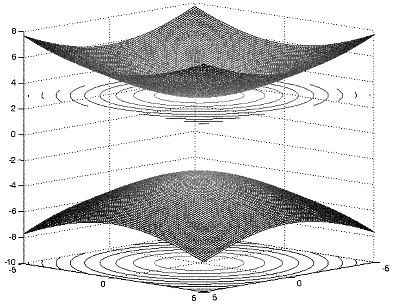
a) xoy ravan ili

b) yoz ravan ili

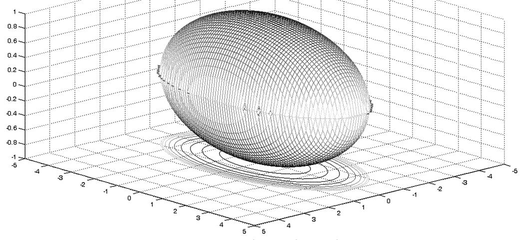
c) xoz ravan

U gornjem primjeru Ω smo $\overset{\text{prvo}}{\vee}$ projicirali na xoy ravan.

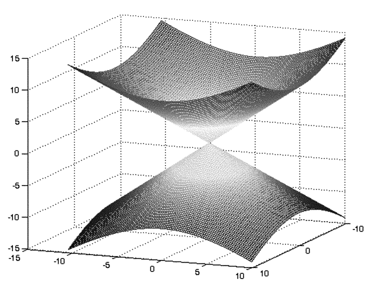
I se može izraziti na 6 načina.



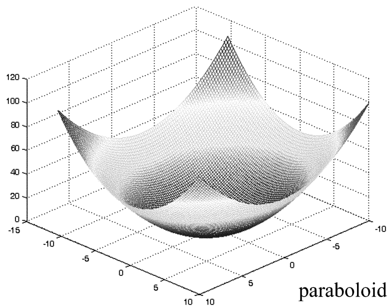
hiperboloid $x^2+y^2-z^2=-9$



elipsoid $\frac{x^2}{4}+\frac{y^2}{16}+\frac{z^2}{9}=1$



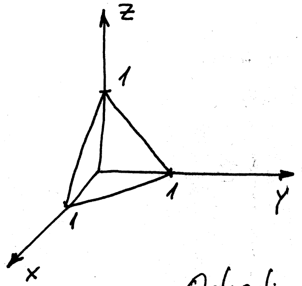
čunj $x^2+y^2=z^2$



paraboloid $2z=x^2+y^2$

Ⓧ Izračunajte $\iiint_{\Omega}(1-x)yz\,dx\,dy\,dz$ gdje je Ω oblast ograničena ravnima $x=0,\ y=0,\ z=0$ i $x+y+z=1$

Rj.



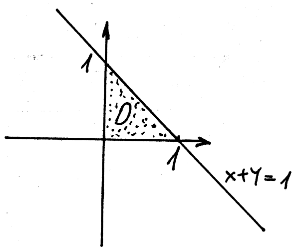
$x+y+z=1 \Leftrightarrow \frac{x}{1}+\frac{y}{1}+\frac{z}{1}=1$ segmentni oblik jedn. ravni

$x=0$ je yoz ravan

$y=0$ je xoz ravan

$z=0$ je xoy ravan

Odredimo projekciju oblasti na xoy ravan



$$\begin{array}{l} x+y+z=1\\ \quad z=0\\ \hline x+y=1\end{array}$$

$z=1-x-y$

Sa slike odredimo granice

$0\le x\le 1$

$0\le y\le 1-x$

$0\le z\le 1-x-y$

$$\iiint_{\Omega}(1-x)yz\,dx\,dy\,dz=\int_0^1(1-x)\,dx\int_0^{1-x}y\,dy\int_0^{1-x-y}z\,dz=\int_0^1(1-x)\,dx\int_0^{1-x}y\,\frac{1}{2}z^2\Big|_0^{1-x-y}dy$$

$$=\frac{1}{2}\int_0^1(1-x)\,dx\int_0^{1-x}y\,(\underbrace{1-x}_{a}-\underbrace{y}_{b})^2\,dy=\frac{1}{2}\int_0^1(1-x)\,dx\int_0^{1-x}y\left[(1-x)^2-2y(1-x)+y^2\right]dy$$

$$=\frac{1}{2}\int_0^1(1-x)\,dx\int_0^{1-x}\left[(1-x)^2y-2y^2(1-x)+y^3\right]dy=\frac{1}{2}\int_0^1(1-x)\left[(1-x)^2\frac{1}{2}y^2\Big|_0^{1-x}-\right.$$

$$\left.-2\cdot\frac{1}{3}y^3\Big|_0^{1-x}\cdot(1-x)+\frac{1}{4}y^4\Big|_0^{1-x}\right]dx=\frac{1}{2}\int_0^1(1-x)\left[(1-x)^4\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)\right]dx$$

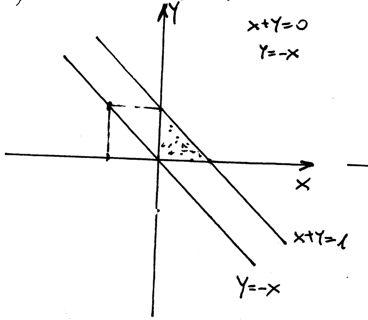
$$=\frac{1}{2}\cdot\frac{1}{12}\int_0^1(1-x)^5\,dx=\left|\begin{array}{ll}1-x=t & x=0\Rightarrow t=1\\ -dx=dt & x=1\Rightarrow t=0\\ dx=-dt & \end{array}\right|=\frac{-1}{24}\int_1^0 t^5\,dt=-\frac{1}{24}\cdot\frac{1}{6}t^6\Big|_1^0=\frac{1}{144}$$

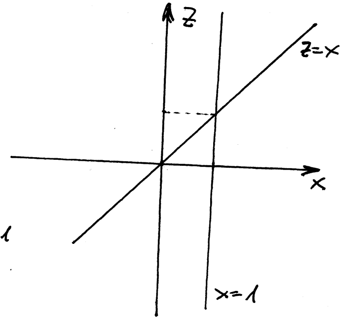
(#) Izračunati trojni integral

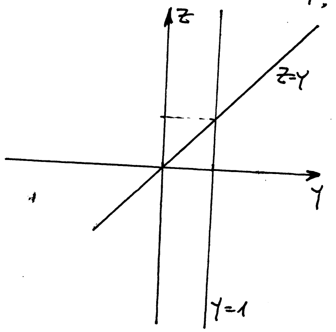
$$I=\iiint_G \frac{1}{(1+z)^3}\,dxdydz$$

gdje je oblast G u I oktantu ograničena ravnima $x+y=1$, $z=x+y$, $x=0$, $y=0$, $z=0$,

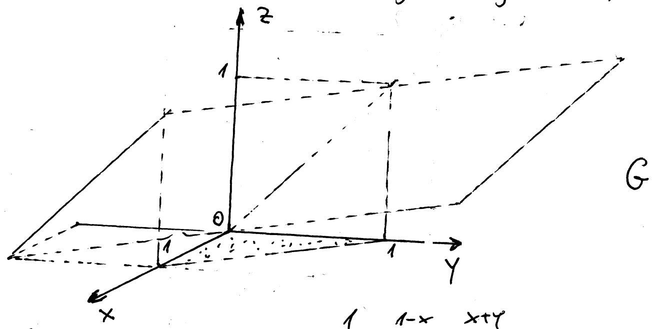
Rj. Napravimo presjek ravni $x+y=1$ i $z=x+y$ sa xOy, xOz i yOz ravni.







Iz presjeka vidimo da je ravan $x+y=1$ paralelna sa z osom a da je oblast G odozgo ograničena sa $z=x+y$ ravni



$$G:\begin{cases}0\leq x\leq 1\\ 0\leq y\leq 1-x\\ 0\leq z\leq x+y\end{cases}$$

$$I=\iiint_G\frac{1}{(1+z)^2}\,dxdydz=\int_0^1 dx\int_0^{1-x}dy\int_0^{x+y}(1+z)^{-3}d(1+z)=\frac{1}{-2}\int_0^1 dx\int_0^{1-x}\overbrace{(1+z)^{-2}\Big|_0^{x+y}}^{(1+(x+y))^{-2}-1^{-2}}dy=$$

$$=-\frac{1}{2}\int_0^1 dx\int_0^{1-x}(1+x+y)^{-2}d(1+x+y)-\left(-\frac{1}{2}\right)\int_0^1 dx\int_0^{1-x}dy=-\frac{1}{2}\int_0^1(-1)(1+x+y)^{-1}\Big|_0^{1-x}dx+$$

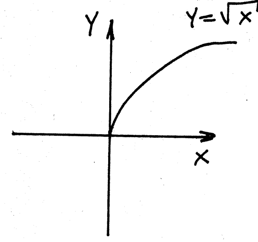
$$+\frac{1}{2}\int_0^1(1-x)\,dx=\frac{1}{2}\int_0^1\underbrace{\left(2^{-1}-(1+x)^{-1}\right)}_{\frac{1}{2}-(1+x)^{-1}}dx+\frac{1}{2}\int_0^1(1-x)\,dx=$$

$$=\frac{1}{2}\left(\frac{1}{2}x\Big|_0^1-\ln(1+x)\Big|_0^1+x\Big|_0^1-\frac{1}{2}x^2\Big|_0^1\right)=$$

$$=\frac{1}{2}\left(\frac{1}{2}-\ln 2+1-\frac{1}{2}\right)=\frac{1}{2}(1-\ln 2)$$ traženo rješenje

(#) Izračunati $I=\iiint\limits_{\Omega} y\cos(x+z)\,dx\,dy\,dz$ gdje je Ω oblast ograničena plohom $y=\sqrt{x}$ i ravnima $y=0$, $z=0$ i $x+z=\frac{\pi}{2}$.

Rj.

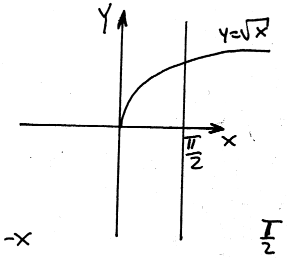


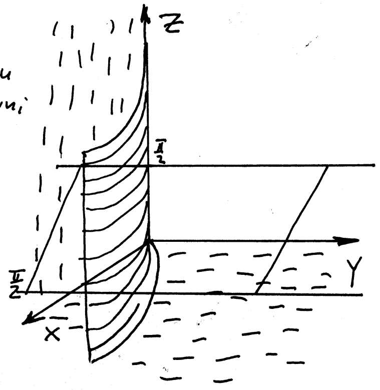
$y=0$ je xOz ravan

$z=0$ je xOy ravan

$x+z=\frac{\pi}{2}$

za $z=0$ dobiću projekciju ove ravni na xoy ravan





$$0\le x\le\frac{\pi}{2}$$
$$0\le y\le\sqrt{x}$$
$$0\le z\le\frac{\pi}{2}-x$$

$$I=\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y\,dy\int_0^{\frac{\pi}{2}-x}\cos(x+z)\,dz=\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y\sin(x+z)\Big|_{z=0}^{z=\frac{\pi}{2}-x}dy\overset{(*)}{=}$$

$$\int\cos(x+a)\,dx=\left|\begin{matrix}x+a=t\\ dx=dt\end{matrix}\right|=\int\cos t\,dt=\sin t+c=\sin(x+a)+c$$

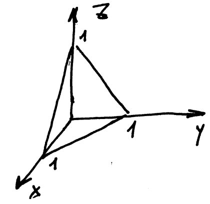
$$\overset{(*)}{=}\int_0^{\frac{\pi}{2}}dx\int_0^{\sqrt{x}}y\Big[\underbrace{\sin(x+\tfrac{\pi}{2}-x)}_{1}-\sin x\Big]dy=\int_0^{\frac{\pi}{2}}\underbrace{\frac{1}{2}y^2\Big|_0^{\sqrt{x}}}_{x-0}(1-\sin x)\,dx=$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}x\,dx-\frac{1}{2}\int_0^{\frac{\pi}{2}}x\sin x\,dx=\left|\begin{matrix}u=x & dv=\sin x\,dx\\ du=dx & v=-\cos x\end{matrix}\right|=\frac{1}{2}\cdot\frac{1}{2}x^2\Big|_0^{\frac{\pi}{2}}-$$

$$-\frac{1}{2}\Big[-x\cos x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x\,dy\Big]=\frac{1}{4}\cdot\frac{\pi^2}{4}-\frac{1}{2}\sin x\Big|_0^{\frac{\pi}{2}}=\frac{\pi^2-8}{16}$$

(#) Izračunati trostruki integral $I=\iiint\limits_{\Omega}\frac{dx\,dy\,dz}{(x+y+z+1)^3}$, ako je Ω oblast omeđena koordinatnim ravnima i ravni $x+y+z=1$.

Rj. $x+y+z=1$ je ravan koja na koordinatnim osama prolazi kroz tačke $(1,0,0)$, $(0,1,0)$ i $(0,0,1)$



$$\Omega:\begin{cases}0\le x\le 1\\ 0\le y\le 1-x\\ 0\le z\le 1-x-y\end{cases}$$

$$I=\int_0^1 dx\int_0^{1-x}dy\int_0^{1-x-y}\frac{dz}{(x+y+z+1)^3}\overset{(*)}{=}$$

$$\int\frac{dz}{(x+y+z+1)^3}=\left|\begin{matrix}x+y+z+1=t\\ dz=dt\end{matrix}\right|=\int\frac{dt}{t^3}=\int t^{-3}dt=\frac{t^{-2}}{-2}+c=$$

$$=\frac{-1}{2(x+y+z+1)^2}+c$$

$$\overset{(*)}{=}\int_0^1 dx\int_0^{1-x}\frac{-1}{2(x+y+z+1)^2}\Big|_0^{1-x-y}dy=\int_0^1 dx\int_0^{1-x}\Big(\frac{-1}{2(x+y+1-x-y+1)^2}-$$

$$-\frac{-1}{2(x+y+0+1)^2}\Big)dy=\int_0^1 dx\int_0^{1-x}\Big(-\frac{1}{8}+\frac{1}{2(x+y+1)^2}\Big)dy=$$

$$=-\frac{1}{2}\int_0^1 dx\int_0^{1-x}\Big(\frac{1}{4}-\frac{1}{(x+y+1)^2}\Big)dy\overset{(**)}{=}-\frac{1}{2}\int_0^1\Big(\frac{1}{4}y\Big|_0^{1-x}+\frac{1}{x+y+1}\Big|_0^{1-x}\Big)dx$$

$$\int\frac{dy}{(x+y+1)^2}=\left|\begin{matrix}x+y+1=t\\ dy=dt\end{matrix}\right|=\int\frac{dt}{t^2}=\frac{t^{-1}}{-1}+c=\frac{-1}{t}+c=\frac{-1}{x+y+1}+c\quad\ldots(**)$$

$$=-\frac{1}{2}\int_0^1\Big(\frac{1}{4}(1-x)+\frac{1}{2}-\frac{1}{x+1}\Big)dx=-\frac{1}{2}\Big(\overbrace{\frac{1}{4}x\Big|_0^1-\frac{1}{4}\cdot\frac{x^2}{2}\Big|_0^1}^{\frac{1}{8}}+\frac{1}{2}x\Big|_0^1-\underbrace{\ln|x+1|\Big|_0^1}_{\ln 2-\ln 1}\Big)=\frac{1}{2}\ln 2-\frac{5}{16}$$

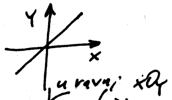
(#) Izračunati trostruki integral $I=\iiint\limits_{\Omega} z\,dx\,dy\,dz$, ako je

$\Omega: y=x,\ y=2x,\ 2x=1,\ x^2+y^2+z^2=1,\ z\geq 0$

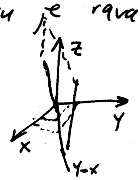
(oblast Ω je ograničena ovim površinama).

Rj. Komentarišimo površi koje čine Ω.

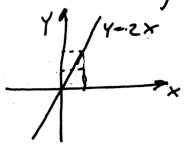
- $y=x$ u ravni je prava



$y=x$ u prostoru je ravan koja sadrži pravu $y=x$

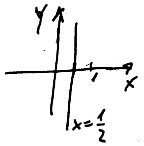


$y=2x$ u ravni je prava

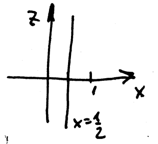


$y=2x$ u prostoru je ravan koja u ravni xOy sadrži pravu $y=2x$

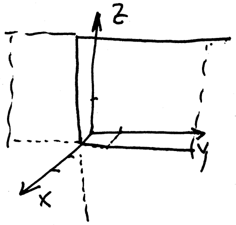
$2x=1$ u ravni xOy je prava



u ravni xOz je isto prava



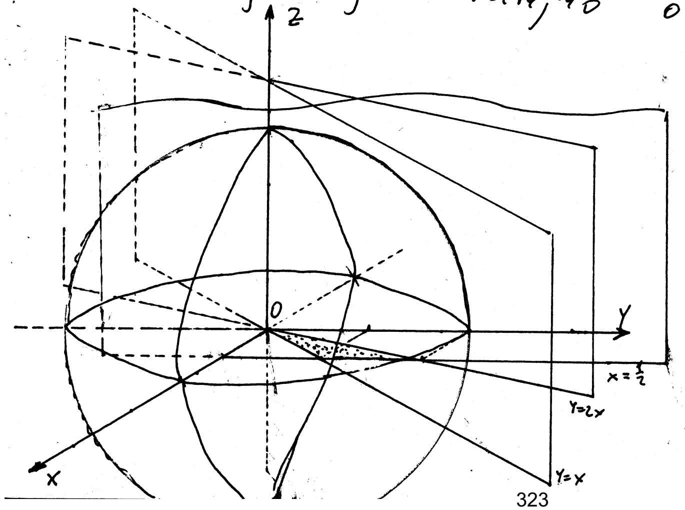
U prostoru to je ravan koja sadrži u xOz ravni pravu $x=\frac{1}{2}$ i u xOY ravni pravu $x=\frac{1}{2}$



$x=\frac{1}{2}$ je ravan koja je paralelna sa YOz osom

$x^2+y^2+z^2=1$ je jednačina kružnice

Na osnovu svega ovoga skicirajmo oblast Ω.



Oblast Ω je kružni isječak čija projekcija na xOy ravan je predstavljena tačkama na slici. Možemo zaključiti

$$\Omega: \begin{cases} 0\leq x\leq \frac{1}{2} \\ x\leq y\leq 2x \\ 0\leq z\leq \sqrt{1-x^2-y^2} \end{cases}$$

$$I=\iiint\limits_{\Omega} z\,dx\,dy\,dz=\int_0^{\frac{1}{2}}dx\int_x^{2x}dy\int_0^{\sqrt{1-x^2-y^2}}z\,dz=\int_0^{\frac{1}{2}}dx\int_x^{2x}\frac{1}{2}z^2\Big|_0^{\sqrt{1-x^2-y^2}}dy=$$

$$=\frac{1}{2}\int_0^{\frac{1}{2}}dx\int_x^{2x}(1-x^2-y^2)\,dy=\frac{1}{2}\int_0^{\frac{1}{2}}\left(y\Big|_x^{2x}-x^2y\Big|_x^{2x}-\frac{1}{3}y^3\Big|_x^{2x}\right)dx=$$

$$=\frac{1}{2}\int_0^{1/2}\left(x-x^3-\frac{1}{3}7x^3\right)dx=\frac{1}{2}\int_0^{1/2}\left(x-\frac{10}{3}x^3\right)dx=\frac{1}{2}\left(\frac{1}{2}x^2\Big|_0^{\frac{1}{2}}-\frac{\overset{5}{\cancel{10}}}{3}\cdot\frac{1}{\underset{2}{\cancel{4}}}x^4\Big|_0^{\frac{1}{2}}\right)$$

$$=\frac{1}{2}\left(\frac{1}{2}\cdot\frac{1}{4}-\frac{5}{6}\cdot\frac{1}{16}\right)=\frac{1}{2}\left(\frac{1}{8}-\frac{5}{}\right)=\frac{1}{2}\cdot\frac{12-5}{96}=\frac{7}{192}$$

1. Izračunaj trostruki integral $I = \int\limits_{-1}^{1} dx \int\limits_{x^2}^{1} dy \int\limits_{0}^{2} (4+z)dz$.

Rješenje:

$$I = \int\limits_{-1}^{1} dx \int\limits_{x^2}^{1} dy \int\limits_{0}^{2} (4+z)dz = \int\limits_{-1}^{1} dx \int\limits_{x^2}^{1} 4z\Big|_{0}^{2} + \frac{z^2}{2}\Big|_{0}^{2} dy = \int\limits_{-1}^{1} dx \int\limits_{x^2}^{1} (8+2)dy = 10 \int\limits_{-1}^{1} y\Big|_{x^2}^{1} dx =$$

$$= 10\int\limits_{-1}^{1}(1-x^2)dx = 10(x\Big|_{-1}^{1} - \frac{x^3}{3}\Big|_{-1}^{1}) = 10(2-\frac{2}{3}) = 10\cdot\frac{4}{3} = \frac{40}{3}$$

2. Izračunaj trostruki integral $\iiint\limits_G \frac{dxdydz}{1-x-y}$, gdje je G ograničena ravnima :

a) x+y+z=1, x=0, y=0, z=0;

b) x=0, x=1, y=2, y=5, z=2, z=4.

Rješenja:

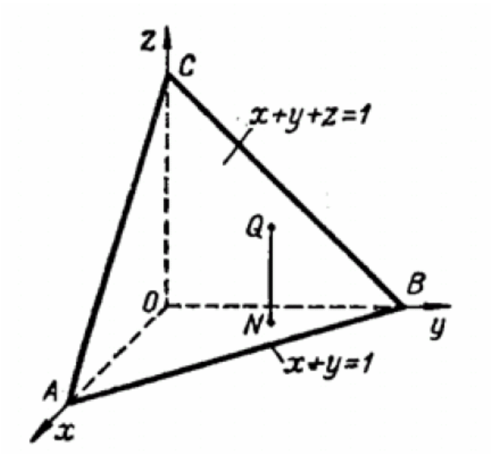
a) $\iiint\limits_G \frac{dxdydz}{1-x-y}$ x=0, y=0, z=0

Skicirajmo oblast G (vidi sliku desno).

x+y+z=1 $\Leftrightarrow \frac{x}{1}+\frac{y}{1}+\frac{z}{1}=1$

x= 0 je yOz ravan
y= 0 je xOz ravan
z= 0 je xOy ravan

Odredimo projekciju oblasti na xOy ravan:
Nacrtati sliku (uputa: pogledati xoy ravan sa slike desno).



x+y+z=1
z= 0

x+y=1
z= 1-x-y

Sa slike projekcije odredimo granice:
$0 \le x \le 1$
$0 \le y \le 1-x$
$0 \le z \le 1-x-y$

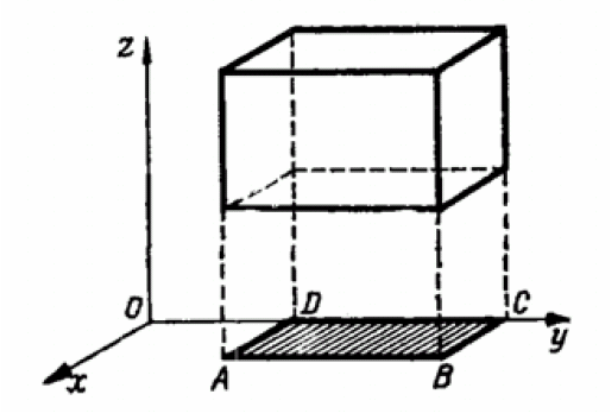
$$\iiint\limits_G \frac{dxdydz}{1-x-y} = \int\limits_0^1 dx \int\limits_0^{1-x} \frac{dy}{1-x-y} \int\limits_0^{1-x-y} dz = \int\limits_0^1 dx \int\limits_0^{1-x} \left(\frac{1}{1-x-y}\cdot z\Big|_0^{1-x-y}\right) dy =$$

$$= \int\limits_0^1 dx \int\limits_0^{1-x} (\frac{1}{1-x-y}\cdot(1-x-y))dy = \int\limits_0^1 dx \int\limits_0^{1-x} dy = \int\limits_0^1 y\Big|_0^{1-x} dx = \int\limits_0^1 (1-x)dx =$$

$$= x\Big|_0^1 - \frac{x^2}{2}\Big|_0^1 = 1-\frac{1}{2} = \frac{1}{2}$$

b) $\iiint\limits_G \frac{dxdydz}{1-x-y}$ x=0, x=1, y=2, y=5, z=2, z=4.

Skicirajmo oblast G (vidi sliku).



$$\int\limits_0^1 dx \int\limits_2^5 \frac{dy}{1-x-y} \int\limits_2^4 dz = \int\limits_0^1 dx \int\limits_2^5 z\Big|_2^4 \frac{dy}{1-x-y} = 2\int\limits_0^1 dx \int\limits_2^5 \frac{dy}{1-x-y} = \left|\begin{array}{l} 1-x-y=t \\ -dy=dt \\ y=2 \Rightarrow t=-1-x \\ y=5 \Rightarrow t=-4-x \end{array}\right| =$$

$$-2\int\limits_0^1 dx \int\limits_{-1-x}^{-4-x} \frac{dt}{t} = -2\int\limits_0^1 \ln|t|\Big|_{-1-x}^{-4-x} = -2\int\limits_0^1 \{\ \ln[-(4-x)] - \ln(-1-x)\ \}dx =$$

$$= -2\int\limits_0^1 \ln|x+4|dx + 2\int\limits_0^1 \ln|x+1|dx \overset{*}{=}$$

Zadaci za vježbu

U zadacima 3474. — 3476. proceniti date integrale.

3474. $\iiint\limits_{\Omega} (x^2+y^2+z^2)\,dv$, gde je Ω—lopta $x^2+y^2+z^2<R^2$.

3475. $\iiint\limits_{\Omega} (x+y+z)\,dv$, gde je Ω—lopta $x \geqslant 1,\ y>1,\ z>1,\ x<3,$ $y<3,\ z<3.$

3476. $\iiint\limits_{\Omega} (x+y-z+10)\,dv$, gde je Ω—lopta $x^2+y^2+z^2<3$.

U zadacima 3517 — 3524 izračunati navedene trostruke i trojne integrale

3517. $\int\limits_0^1 dx \int\limits_0^2 dy \int\limits_0^3 dz.$ **3518.** $\int\limits_0^a dx \int\limits_0^b dy \int\limits_0^c (x+y+z)\,dz.$

3519. $\int\limits_0^a dx \int\limits_0^x dx \int\limits_0^y xyz\,dz.$ **3520.** $\int\limits_0^a dx \int\limits_0^x dy \int\limits_0^{xy} x^3y^2z\,dz.$

3521. $\int\limits_0^{e-1} dx \int\limits_0^{e-x-1} dy \int\limits_e^{x+y+e} \frac{\ln(z-x-y)}{(x-e)(x+y-e)}\,dz.$

3522. $\iiint\limits_{\Omega} \frac{dx\,dy\,dz}{(x+y+z+1)^3}$, Ω je oblast ograničena ravnima $x=0,\ y=0,$ $z=0,\ x+y+z=1.$

3523. $\iiint\limits_{\Omega} xy\,dx\,dy\,dz$, Ω je oblast ograničena hiperboličnim paraboloidom $z=xy$ i ravnima $x+y=1$ i $z=0$ $(z \geq 0)$.

3524. $\iiint\limits_{\Omega} y\cos(z+x)\,dx\,dy\,ds$, Ω je oblast ograničena cilindrom $y=\sqrt{x}$ i ravnima $y=0,\ z=0$ i $x+z=\frac{\pi}{2}$.

Rješenja

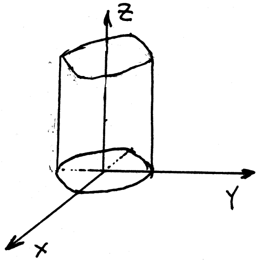
3474. $0<I<\frac{4}{3}\pi R^5$. **3475.** $24<I<72$.

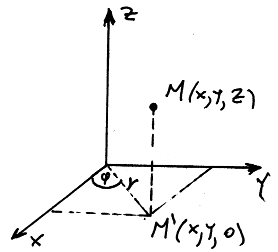
3476. $29\pi\sqrt{3}<I<52\pi\sqrt{3}$. **3517.** 6. **3518.** $\frac{abc(a+b+c)}{2}$. **3519.** $\frac{a^6}{48}$. **3520.** $\frac{a^{11}}{110}$.

3521. $2e-5$. **3522.** $\frac{1}{2}\left(\ln 2-\frac{5}{8}\right)$. **3523.** $\frac{1}{180}$. **3524.** $\frac{\pi^2}{16}-\frac{1}{2}$.

Računanje trostrukih integrala uvođenjem cilindričnih i sfernih koordinata

cilindrične koordinate





uvodimo smjenu

$x = r\cos\varphi$

$y = r\sin\varphi$

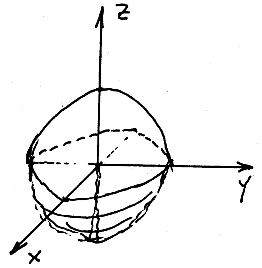
$z = z$

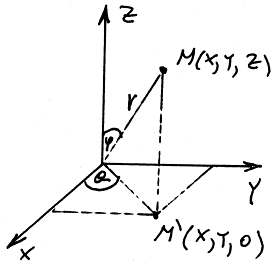
$dx\,dy\,dz = r\,dr\,d\varphi\,dz$

cilindrične koordinate obično uvedemo ako se pojavi izraz x^2+y^2 $\left(x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2(\sin^2\varphi+\cos^2\varphi) = r^2\right)$

$(r \geq 0,\quad 0 \leq \varphi \leq 2\pi)$

sferne koordinate





uvodimo smjenu

$x = r\sin\varphi\cos\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi\,dr\,d\varphi\,d\theta$

$r \geq 0$

$0 \leq \varphi \leq \pi$

$0 \leq \theta \leq 2\pi$

$x^2+y^2+z^2 = r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\varphi\sin^2\theta +$
$+ r^2\cos^2\varphi = \ldots = r^2$

sferne koordinate obično uvodimo ako se u podintegralnoj f-ji ili u opisu oblasti integracije pojavljuje izraz $x^2+y^2+z^2$.

Dati trojni integral $\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz$

transformisati na trostruki u cilindričnim koordinatama (sa određenim posebnim granicama integracije) ako je Ω oblast u prvom oktantu ograničen cilindrom $x^2+y^2=R^2$ i ravnima $z=0$, $z=1$, $y=x$ i $y=x\sqrt{3}$

Rj.

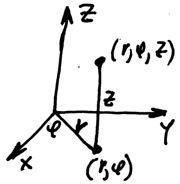
Cilindrične koordinate glase

$x = r\cos\varphi$

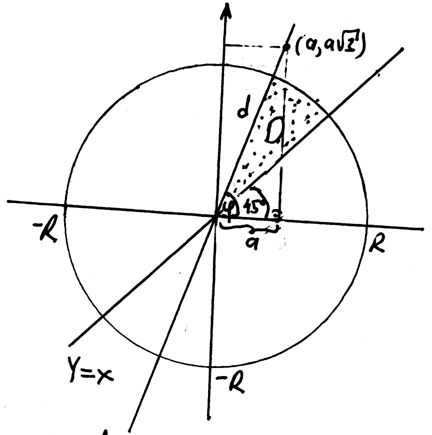
$y = r\sin\varphi$

$z = z$

$dx\,dy\,dz = r\,dr\,d\varphi\,dz$



Napravimo presjek datih površina sa xOy ravni:



$$\cos\varphi = \frac{a}{d} = \frac{a}{2a} = \frac{1}{2} \Rightarrow \varphi = \frac{\pi}{3}$$

$$d^2 = a^2 + 3a^2 = 4a^2$$

$$d = 2a$$

Sad nije teško vidjeti da je

$$\iiint\limits_{\Omega} f(x,y,z)\,dx\,dy\,dz = \int_0^1 dz \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^R f(r\cos\varphi, r\sin\varphi, z)\, r\,dr.$$

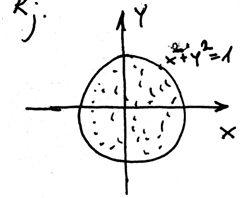
$$\Omega \xrightarrow{\text{transformiše}} \Omega'$$

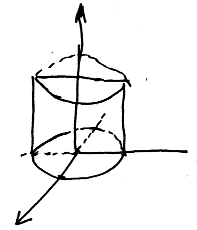
$$\Omega' : \begin{cases} 0 \le r \le R \\ \frac{\pi}{4} \le \varphi \le \frac{\pi}{3} \\ 0 \le z \le 1 \end{cases}$$

Izračunati $I = \iiint\limits_{\Omega} (x^2+y^2+z)^3\,dx\,dy\,dz$ gdje je

Ω oblast ograničena sa $x^2+y^2=1$, $z=0$ i $z=1$.

Rj.





uvodimo smjene:

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$

$x^2+y^2=1$

$r^2=1$

$r \ge 0$

$0 \le r \le 1$

$0 \le z \le 1$

$0 \le \varphi \le 2\pi$

$$\Omega' = \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le 2\pi \\ 0 \le z \le 1 \end{cases}$$

$dx\,dy\,dz = r\,dr\,d\varphi\,dz$

$$I = \iiint\limits_{\Omega} (x^2+y^2+z)^3\,dx\,dy\,dz = \iiint\limits_{\Omega'} (r^2+z)^3 r\,dr\,d\varphi\,dz =$$

$$= \int_0^1 dz \int_0^{2\pi} d\varphi \int_0^1 (r^2+z)^3 \cdot r\,dr = \left| \begin{matrix} r^2+z=t & r=0 \Rightarrow t=z \\ 2r\,dr = dt & r=1 \Rightarrow t=z+1 \\ r\,dr = \frac{1}{2}dt & \end{matrix} \right|$$

$$= \frac{1}{2}\int_0^1 dz \int_0^{2\pi} d\varphi \int_z^{z+1} t^3\,dt = \frac{1}{2}\int_0^1 dz \int_0^{2\pi} \frac{1}{4} t^4 \Big|_z^{z+1} d\varphi = \frac{1}{8}\int_0^1 \left[(z+1)^4 - z^4\right] \varphi \Big|_0^{2\pi} dz$$

$$= \frac{1}{8} \cdot 2\pi \int_0^1 \left[(z+1)^4 - z^4\right] dz = \frac{\pi}{4} \cdot \left(\frac{1}{5}(z+1)^5 \Big|_0^1 - \frac{1}{5} z^5 \Big|_0^1 \right) =$$

$$\int (z+1)^4\,dz = \left| \begin{matrix} t = z+1 \\ dt = dz \end{matrix} \right| = \int t^4\,dt = \frac{1}{5}t^5 + C = \frac{1}{5}(z+1)^5 + C$$

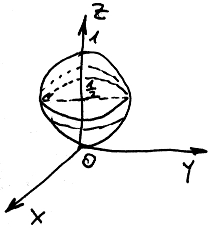
$$= \frac{\pi}{20}(31 - 1) = \frac{30\pi}{20} = \frac{3\pi}{2}$$

(#) Izračunati $$I = \iiint_{\Omega} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz$$ gdje je Ω oblast ograničena sferom $x^2+y^2+z^2 = z$.

Rj.

$$x^2+y^2+z^2 = z$$
$$x^2+y^2+z^2 - z = 0$$
$$x^2+y^2+z^2 - 2\cdot z\cdot\frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = 0$$
$$x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \frac{1}{4}$$

centar sfere u tački $S(0,0,\frac{1}{4})$
polupréčnik sfere $r = \frac{1}{2}$



uvodimo smjene

$$x = r\sin\varphi\cos\theta$$
$$y = r\sin\varphi\sin\theta$$
$$z = r\cos\varphi$$

Odredimo granice za r, φ i θ nove oblasti

$$x^2+y^2+z^2 = r^2$$

$r^2 = r\cos\varphi$ ↗ iz $x^2+y^2+z^2 = z$

$/:r \quad (r \neq 0)$

$r = \cos\varphi$ kako je $r > 0 \Rightarrow \cos\varphi > 0$ tj. $0 \le \varphi \le \frac{\pi}{2}$

$$\Omega' : \begin{cases} 0 \le r \le \cos\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le \theta \le 2\pi \end{cases} \qquad dx\,dy\,dz = r^2\sin\varphi\,dr\,d\varphi\,d\theta$$

$$I = \iiint_{\Omega} \sqrt{x^2+y^2+z^2}\,dx\,dy\,dz = \iiint_{\Omega'} \sqrt{r^2}\,r^2\sin\varphi\,dr\,d\varphi\,d\theta =$$

$$\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \sin\varphi\,d\varphi \int_0^{\cos\varphi} r^3\,dr = \int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\varphi\,\frac{1}{4}r^4\Big|_0^{\cos\varphi} d\varphi =$$

$$= \frac{1}{4}\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\varphi\cos^4\varphi\,d\varphi = \left| \begin{array}{ll} \cos\varphi = t & \varphi = 0 \Rightarrow t = 1 \\ -\sin\varphi\,d\varphi = dt & \varphi = \frac{\pi}{2} \Rightarrow t = 0 \\ \sin\varphi\,d\varphi = -dt & \end{array} \right| =$$

$$= \frac{1}{4}\int_0^{2\pi} d\theta \int_0^{1} t^4\,dt = \frac{1}{4}\int_0^{2\pi} \frac{1}{5}t^5\Big|_0^1 d\theta = \frac{1}{20}\,\theta\Big|_0^{2\pi} = \frac{1}{20}\cdot 2\pi = \frac{\pi}{10}$$

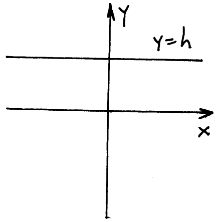
(#) Izračunati trostruki integral

$$K = \iiint_{T} y\,dx\,dy\,dz$$

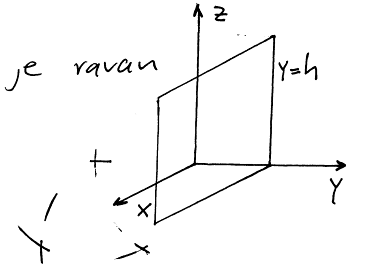
gdje je oblast T ograničena površinama $y = \sqrt{x^2+z^2}$ i $y = h$, $h > 0$.

Rj. Pokušajmo skicirati oblast T.

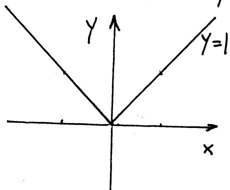
U xOy-ravni $y = h$ je prava.



U prostoru $y = h$ je ravan

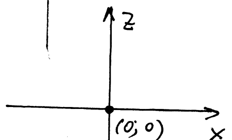


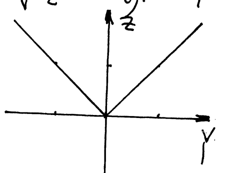
U xOy-ravni površina $y = \sqrt{x^2+z^2}$ je oblika $y = \sqrt{x^2}$



U xOz-ravni površina $y = \sqrt{x^2+z^2}$ je oblika $0 = \sqrt{x^2+z^2}$ tj. tačka $(0;0)$.

U yOz-ravni površina $y = \sqrt{x^2+z^2}$ je oblika $y = \sqrt{z^2}$ tj. $y = |z|$.

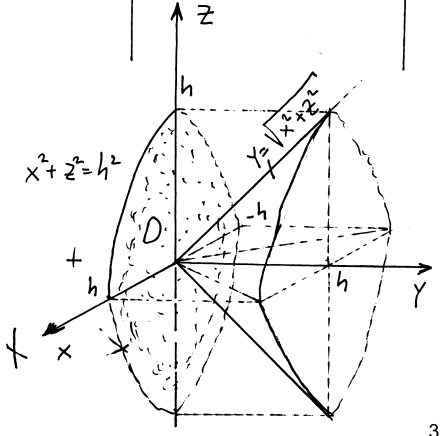




Ako napravimo presjek površina $y = \sqrt{x^2+z^2}$ i $y = h$ dobićemo $h = \sqrt{x^2+z^2}$ tj.

$$x^2+z^2 = h^2$$

(krug polupréčnika h)



Oblast T (pola čunja) je prikazan na slici lijevo. Ako napravimo projekciju oblasti T na xOz ravan dobićemo sljedeće granice:

$$T:\begin{cases}-h \le x \le h\\ -\sqrt{h^2-z^2} \le z \le \sqrt{h^2-z^2}\\ \sqrt{x^2+z^2} \le y \le h\end{cases}$$

Pomoću pravougaonih koordinata dati trostruki integral je teško izračunati.

Uvodimo cilindrične koordinate i to

$$x = r\cos\varphi$$
$$z = r\sin\varphi$$
$$y = y$$
$$dx\,dy\,dz = r\,dr\,d\varphi\,dy$$

$$T \xrightarrow{\text{transformiše}} T':\begin{cases}0 \le r \le h\\ 0 \le \varphi \le 2\pi\\ r \le y \le h\end{cases}$$

Prema tome

$$K = \iiint\limits_T y\,dx\,dy\,dz = \left|\begin{matrix}\text{uvodimo}\\ \text{cilindrične}\\ \text{koordinate}\end{matrix}\right| = \iiint\limits_{T'} y\,r\,dr\,d\varphi\,dy =$$

$$= \int_0^{2\pi} d\varphi \int_0^h r\,dr \int_r^h y\,dy = \int_0^{2\pi} d\varphi \int_0^h r\,\underbrace{\frac{1}{2}y^2\Big|_r^h}_{h^2-r^2} dr =$$

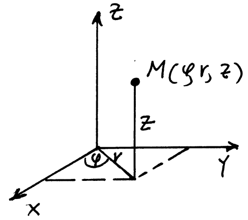
$$= \frac{1}{2}\int_0^{2\pi} d\varphi \int_0^h (rh^2 - r^3)\,dr = \frac{1}{2}\int_0^{2\pi}\Big(\underbrace{\frac{1}{2}r^2h^2\Big|_0^h}_{\frac{1}{2}r^4} - \underbrace{\frac{1}{4}r^4\Big|_0^h}_{-\frac{1}{4}h^4}\Big)\,d\varphi$$

$$= \frac{1}{2}\cdot\frac{1}{4}h^4\int_0^{2\pi} d\varphi = \frac{1}{8}h^4\,\varphi\Big|_0^{2\pi} = \frac{h^4\pi}{4}$$

traženo rješenje

(#) Dat je trostruki integral $\int_0^{2\pi} d\varphi \int_0^2 r^3\,dr \int_0^{\sqrt{4-r^2}} dz$ u cilindričnim koordinatama. Skicirati oblast integracije i izračunati taj integral prelazeći na sferne koordinate.

Rj: U cilindričnim koordinatama proizvoljna tačka M je opisana na sljedeći način



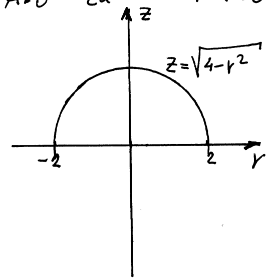
$$\Omega:\begin{cases}0 \le \varphi \le 2\pi\\ 0 \le r \le 2\\ 0 \le z\sqrt{4-r^2}\end{cases}$$

Na osnovu izgleda oblasti Ω vidimo da je projekcija figure na xOy ravan oblika

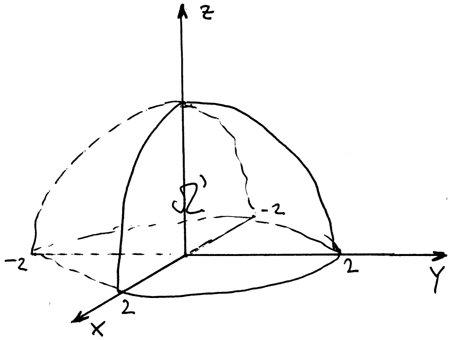
$$D:\begin{cases}0 \le \varphi \le 2\pi\\ 0 \le r \le 2\end{cases}$$

tj. krug sa centrom u koordinatnom početku poluprečnika 2.

Ako za fiksirano φ posmatramo rOz ravan imamo



Prema tome oblast integracije Ω je polulopta



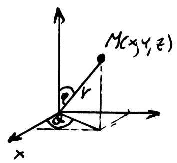
Cilindrične koordinate glase

$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = z$$
$$dx\,dy = r\,dr\,d\varphi$$

Tako da bi prelaskom na pravougaone koordinate sad imali

$$\int_0^{2\pi} d\varphi \int_0^2 r^3 dr \int_0^{\sqrt{1-r^2}} dz = \iiint_\Omega r^2\, r\,dr\,d\varphi\, dz = \left| \begin{array}{l} \text{prelazimo na pravougaone koordinate} \\ \Omega \xrightarrow{\text{transformiše}} \Omega' \\ r\,dr\,d\varphi = dx\,dy \\ r^2 = r^2(\sin^2\varphi+\cos^2\varphi) \\ \quad = r^2\sin^2\varphi + r^2\cos^2\varphi \\ \quad = (r\sin\varphi)^2+(r\cos\varphi)^2 \\ \quad = x^2+y^2 \end{array} \right| =$$

$$= \iiint_{\Omega'} (x^2+y^2)\, dx\,dy\,dz$$



Sferne koordinate glase

$x = r\sin\varphi\cos\alpha$

$y = r\sin\varphi\sin\alpha$

$z = r\cos\varphi$

$dx\,dy\,dz = r^2\sin\varphi\, dr\,d\varphi\,d\alpha$

$$\Omega' \xrightarrow{\text{transformiše}} \Omega'' : \begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le \alpha \le 2\pi \end{cases} \qquad x^2+y^2 = r^2\sin^2\varphi$$

$$\iiint_{\Omega'} (x^2+y^2)\,dx\,dy\,dz = \left| \begin{array}{l} \text{uvodimo} \\ \text{sferne} \\ \text{koordinate} \end{array} \right| = \iiint_{\Omega''} r^2\sin^2\varphi\; r^2\sin\varphi\, dr\,d\alpha\,d\varphi =$$

$$= \int_0^{2\pi} d\alpha \int_0^2 r^4 dr \int_0^{\frac{\pi}{2}} \sin^3\varphi\, d\varphi \overset{(*)}{=} \alpha\Big|_0^{2\pi} \cdot \frac{1}{5}r^5\Big|_0^2 \cdot \frac{2}{3} = \frac{2^7}{15}\pi$$

traženo rješenje

$$\int_0^{\frac{\pi}{2}} \sin^3\varphi\, d\varphi = \int_0^{\frac{\pi}{2}} \sin\varphi \underbrace{(1-\cos^2\varphi)}_{\sin^2\varphi} d\varphi = \left| \begin{array}{l} d(\sin\varphi) = \cos\varphi\, d\varphi \\ d(\cos\varphi) = -\sin\varphi\, d\varphi \end{array} \right| = -\int_0^{\frac{\pi}{2}} (1-\cos^2\varphi)\, d(\cos\varphi)$$

$$= -\left(\cos\varphi\Big|_0^{\frac{\pi}{2}} - \frac{1}{3}\cos^3\varphi\Big|_0^{\frac{\pi}{2}} \right) = -\left((0-1) - \frac{1}{3}(0-1)\right) = -\left(-1+\frac{1}{3}\right) = \frac{2}{3} \quad \dots(*)$$

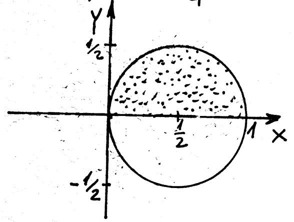
#Izračunati integral $\iiint_\Omega \sqrt{z\cdot(x^2+y^2)}\, dx\,dy\,dz$ gdje je Ω oblast $x^2+y^2 \le x$; $y \ge 0$, $z \ge 0$, $z \le 3$.

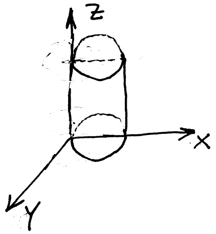
Rj. U ravni xoy kako izgleda $x^2+y^2 \le x$?

$x^2 - x + y^2 = 0$

$x^2 - 2\cdot\frac{1}{2}\cdot x + \frac{1}{4} + y^2 = \frac{1}{4}$

$(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$





Uvodimo smjenu

$x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$x^2+y^2 \le x$

$r^2\cos^2\varphi + r^2\sin^2\varphi \le r\cos\varphi$

$r^2 \le r\cos\varphi \quad /:r \quad (r \ne 0)$

$r \le \cos\varphi$

kako je $r \ge 0$ to je $\cos\varphi \ge 0$

$y \ge 0$

$r\sin\varphi \ge 0 \quad /:r$

$\sin\varphi \ge 0$

imam $0 \le r \le \cos\varphi$

$\sin\varphi \ge 0$

$\cos\varphi \ge 0$

$0 \le z \le 3$

$$\Rightarrow \quad \Omega' : \begin{cases} 0 \le r \le \cos\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le z \le 3 \end{cases} \qquad dx\,dy\,dz = J\,d\varphi\,dr\,dz$$

$$\iiint_\Omega \sqrt{z(x^2+y^2)}\, dx\,dy\,dz = \iiint_{\Omega'} \sqrt{z r^2}\; r\, d\varphi\,dr\,dz = \int_0^3 dz \int_0^{\frac{\pi}{2}} d\varphi \int_0^{\cos\varphi} \sqrt{z}\, r^2 dr =$$

$$= \int_0^3 \sqrt{z}\,dz \int_0^{\frac{\pi}{2}} \frac{1}{3}r^3\Big|_0^{\cos\varphi} d\varphi = \frac{1}{3}\int_0^3 \sqrt{z}\,dz \int_0^{\frac{\pi}{2}} \cos^3\varphi\, d\varphi = \frac{1}{3}\int_0^3 \sqrt{z}\,dz \int_0^{\frac{\pi}{2}} \cos\varphi(1-\sin^2\varphi)\,d\varphi$$

$$= \left| \begin{array}{ll} \sin\varphi = t & \varphi = 0 \Rightarrow t = 0 \\ \cos\varphi\, d\varphi = dt & \varphi = \frac{\pi}{2} \Rightarrow t = 1 \end{array} \right| = \frac{1}{3}\int_0^3 \sqrt{z}\,dz \int_0^1 (1-t^2)\,dt = \frac{1}{3}\int_0^3 \sqrt{z}\left(t\Big|_0^1 - \frac{1}{3}t^3\Big|_0^1\right)dz$$

$$= \frac{1}{3}\left(1-\frac{1}{3}\right) \frac{z^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^3 = \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{2}{3}\; 3^{\frac{3}{2}} = \frac{4}{27}\sqrt{3^3} = \frac{4}{9}\sqrt{3}$$

(#) Izračunati trostruki integral $J=\iiint\limits_{W}(x^2+y^2+z^2)\,dx\,dy\,dz$

gdje je oblast W ograničena površinom $3(x^2+y^2)+z^2=3a^2$.

Rj. Skicirajmo oblast W

$$3(x^2+y^2)+z^2=3a^2$$

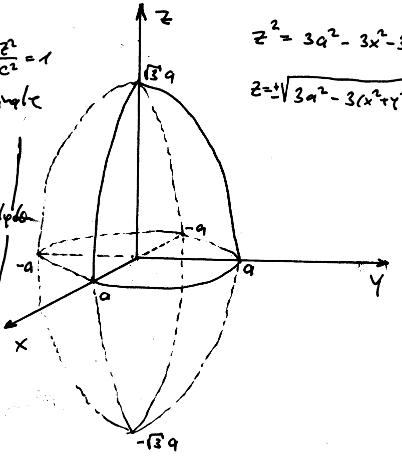
$$3x^2+3y^2+z^2=3a^2 \quad /:3a^2$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}+\frac{z^2}{3a^2}=1$$

jednačina elipse

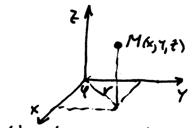
za elipsu $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ uopštene sferne koordinate glase
$x=ar\sin\varphi\cos\theta$
$y=br\sin\varphi\sin\theta$
$z=cr\cos\varphi$
$dxdydz=abcr^2\sin\varphi\,drd\varphi d\theta$

U ovom slučaju uopštene sferne koordinate ne mogu na lagan način rješit zadatak



$$z^2=3a^2-3x^2-3y^2$$

$$z=\pm\sqrt{3a^2-3(x^2+y^2)}$$



Uvodimo cilindrične koordinate

$x=r\cos\varphi$
$y=r\sin\varphi$
$z=z$
$dxdydz=r\,dr\,d\varphi\,dz$

$$x^2+y^2+z^2=r^2+z^2$$

$$W \xrightarrow{\text{transformiše}} W':\begin{cases}0\le r\le a\\ 0\le\varphi\le 2\pi\\ -\sqrt{3(a^2-(x^2+y^2))}\le z\le\sqrt{3(a^2-(x^2+y^2))}\end{cases}$$

$$-\sqrt{3(a^2-r^2)}\le z\le\sqrt{3(a^2-r^2)}$$

$$J=\iiint\limits_{W}(x^2+y^2+z^2)\,dx\,dy\,dz=\left|\begin{matrix}\text{uvodimo cilindrične}\\ \text{koordinate}\end{matrix}\right|=\iiint\limits_{W'}(r^2+z^2)\,r\,dr\,d\varphi\,dz=$$

$$=\int_0^{2\pi}d\varphi\int_0^{a}dr\int_{-\sqrt{3(a^2-r^2)}}^{\sqrt{3(a^2-r^2)}}(r^2+z^2)r\,dz=\int_0^{2\pi}d\varphi\int_0^{a}\left(r^2 z\Big|_{-\sqrt{3(a^2-r^2)}}^{\sqrt{3(a^2-r^2)}}+\frac{z^3}{3}\Big|_{-\sqrt{3(a^2-r^2)}}^{\sqrt{3(a^2-r^2)}}\right)r\,dr$$

$$=\int_0^{2\pi}d\varphi\int_0^{a}\left(r^2\,2\sqrt{3}\sqrt{a^2-r^2}+\frac{1}{3}\left(3\sqrt{3}\sqrt{(a^2-r^2)^3}+3\sqrt{3}\sqrt{(a^2-r^2)^3}\right)\right)r\,dr=$$

ako ovo izvučemo ispred zagrade

$$=\int_0^{2\pi}d\varphi\int_0^{a}\left(2\sqrt{3}\,r^2\underline{\underline{\sqrt{a^2-r^2}}}+2\sqrt{3}\,(a^2-r^2)\underline{\underline{\sqrt{a^2-r^2}}}\right)r\,dr=2\sqrt{3}\,a^2\int_0^{2\pi}d\varphi\int_0^{a}\sqrt{a^2-r^2}\,r\,dr$$

$$=\left|\,d(a^2-r^2)=-2r\,dr\,\right|=-\sqrt{3}\,a^2\int_0^{2\pi}d\varphi\int_0^{a}\sqrt{a^2-r^2}\,d(a^2-r^2)=\;\ldots\;=\frac{1}{\sqrt{3}}4\pi a^5$$

rješenje

(#) Izračunati $I=\iiint\limits_{\Omega}\sqrt{x^2+y^2}\,dx\,dy\,dz$ gdje je Ω oblast

$$x^2+y^2+z^2\le 1 \quad \text{i} \quad x^2+y^2+z^2\le 2z.$$

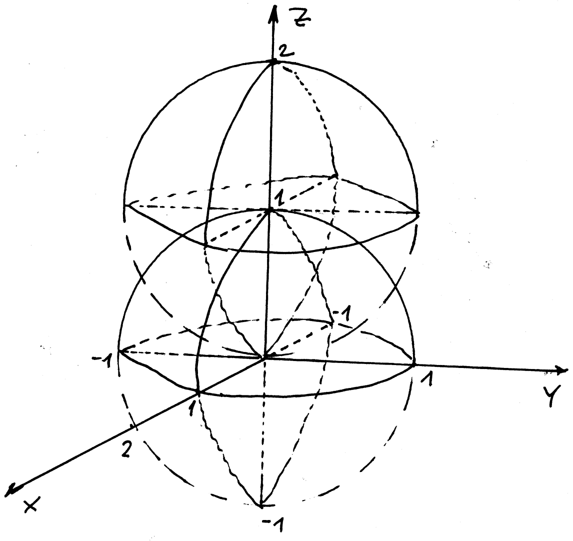
Rj. $x^2+y^2+z^2\le 1$

je unutrašnjost sfere poluprečnika 1 sa centrom u tački (0,0,0)

$x^2+y^2+z^2\le 2z$
$x^2+y^2+z^2-2z\le 0$
$x^2+y^2+z^2-2\cdot z\cdot 1+1\le 1$
$x^2+y^2+(z-1)^2\le 1$

unutrašnjost sfere poluprečnika 1 sa centrom u tački (0,0,1)

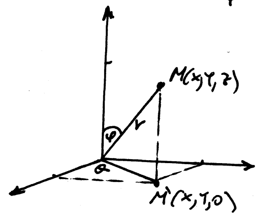
Skicirajmo dvije date sfere



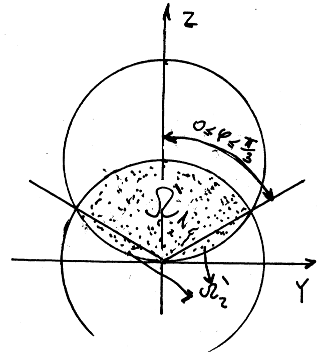
opišimo oblast integracije uz pomoć sfernih koordinata

uvodimo smjenu

$x=r\sin\varphi\cos\theta$
$y=r\sin\varphi\sin\theta$
$z=r\cos\varphi$
$dxdydz=r^2\sin\varphi\,drd\varphi d\theta$



Napravimo projekciju oblasti na yoz ravan.



$x^2+y^2+z^2\le 1$
$(r\sin\varphi\cos\theta)^2+(r\sin\varphi\sin\theta)^2+(r\cos\varphi)^2\le 1$
$r^2\le 1$
$0\le r\le 1$

$x^2+y^2+z^2\le 2z$
$r^2\le 2r\cos\varphi \quad /:r$
$r\le 2\cos\varphi$

$0\le r\le 2\cos\varphi$
$0\le\cos\varphi\le 1 \quad \forall\varphi$

Može biti $2\cos\varphi < 1$ i $2\cos\varphi > 1$.

1° $2\cos\varphi < 1 \Rightarrow \cos\varphi < \frac{1}{2}$ (pa kako je $\cos\varphi > 0$) $\Rightarrow \varphi \in (\frac{\pi}{3}, \frac{\pi}{2})$

$$\Omega_2 : \begin{cases} 0 \le r \le 2\cos\varphi \\ \frac{\pi}{3} \le \varphi \le \frac{\pi}{2} \\ 0 \le \theta \le 2\pi \end{cases}$$

2° $2\cos\varphi > 1 \Rightarrow \cos\varphi > \frac{1}{2}$ (pa kako je $\cos\varphi \le 1$) $\Rightarrow \varphi \in (0, \frac{\pi}{3})$

$$\Omega_1 : \begin{cases} 0 \le r \le 1 \\ 0 \le \varphi \le \frac{\pi}{3} \\ 0 \le \theta \le 2\pi \end{cases}$$

$\Omega = \Omega_1 \cup \Omega_2$ (Ω_1' i Ω_2' su projekcije oblasti Ω na yOz ravan) (vidi sliku)

$$x^2+y^2 = r^2\sin^2\varphi\cos^2\theta + r^2\sin^2\varphi\sin^2\theta = r^2\sin^2\varphi$$

$$I = \iiint\limits_{\Omega} \sqrt{x^2+y^2}\,dx\,dy\,dz = \iiint\limits_{\Omega_1} \sqrt{x^2+y^2}\,dx\,dy\,dz + \iiint\limits_{\Omega_2} \sqrt{x^2+y^2}\,dx\,dy\,dz = I_1 + I_2$$

$$I_1 = \iiint\limits_{\Omega_1} \sqrt{r^2\sin^2\varphi}\; r^2\sin\varphi\,dr\,d\varphi\,d\theta = \int_0^{2\pi} d\theta \int_0^1 r^3\,dr \int_0^{\pi/3} \sin^2\varphi\,d\varphi =$$

$$= \int_0^{2\pi} d\theta \int_0^1 r^3\,dr \int_0^{\pi/3} \frac{1}{2}(1-\cos2\varphi)\,d\varphi = \frac{1}{2}\int_0^{2\pi} d\theta \int_0^1 r^3 \left(\varphi\Big|_0^{\frac{\pi}{3}} - \frac{1}{2}\sin2\varphi\Big|_0^{\frac{\pi}{3}}\right) = \ldots = \frac{\pi\sqrt{3}}{32} + \frac{\pi^2}{12}$$

$$I_2 = \int_0^{2\pi} d\theta \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^2\varphi \int_0^{2\cos\varphi} r^3\,dr = \ldots = \frac{\pi^2}{12} - \frac{\pi\sqrt{3}}{8}$$

$$I = I_1 + I_2 = \frac{2\pi^2}{12} - \frac{3\pi\sqrt{3}}{32} \swarrow \text{traženo rješenje}$$

Zadaci za vježbu

U zadacima 3547 — 3551 transformisati trojni integral $\iiint\limits_{\Omega} f(x, y, z)\,dx\,dy\,dz$ na oilindrične koordinate ρ, φ, z ($x=\rho\cos\varphi$, $y=\rho\sin\varphi, z=z$), ili na sferne kordinate ρ, θ, φ ($x=\rho\cos\varphi\cdot\sin\theta$, $y=\rho\sin\varphi\sin\theta$, $z=\rho\cos\theta$), a zatim ga svesti na trostruki (sa određenim posebnim granicama integracije).

3547. **Ω je oblast u prvom oktantu ograničena cilindrom $x^2+y^2=R^2$ i ravnima $z=0$, $z=1$, $y=x$ i $y=x\sqrt{3}$.**

3548. **Ω je oblast ograničena cilindrom $x^2+y^2=2x$, ravni $s=0$ i paraboloidom $z=x^2+y^2$.**

3549. **Ω je deo lopte $x^2+y^2+z^2<R^2$ koji leži u prvom oktantu.**

3550. **Ω je deo lopte $x^2+y^2+z^2<R^2$ koji leži unutar cilindra $(x^2+y^2)^2 = R^2(x^2-y^2)$ $(x>0)$.**

3551. **Ω je oblast koja predstavlja zajednički deo dve lopte $x^2+y^2+z^2<R^2$ i $x^2+y^2+(z-R)^2<R^2$.**

U zadacima 3552 — 3556 izračunati date integrale prelazeći na cilindrične ili sferne koordinate.

3552. $\displaystyle\int_0^1 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int_0^a dz.$

3553. $\displaystyle\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}\,dz.$

3554. $\displaystyle\int_{-R}^{-R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_0^{\sqrt{R^2-x^2-y^2}} (x^2+y^2)\,zd.$

3555. $\displaystyle\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz.$

3556. $\displaystyle\iiint\limits_{\Omega} (x^2+y^2)\,dx\,dy\,dz.$ gde je oblast Ω određena nejednakostima $z\geqslant 0$, $r^2 \leqslant x^2+y^2+z^2 \leqslant R^2$.

3557. $\displaystyle\iiint\limits_{\Omega} \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+(z-2)^2}}$, gde je Ω—lopta $x^2+y^2+z^2<1$.

3558. $\displaystyle\iiint\limits_{\Omega} \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+(z-2)^2}}$, gde je Ω—cilindar $x^2+y^2<1$, $-1<z<1$.

Rješenja

3547. $\displaystyle\int_0^1 dz \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^R f(\rho\cos\varphi,\ \rho\sin\varphi,\ z)\,\rho\,d\rho.$

3548. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} \rho\,d\rho \int_0^{\rho^2} f(\rho\cos\varphi,\ \rho\sin\varphi.\ z)\,dz.$

3549. $\displaystyle\int_0^{\frac{\pi}{2}} \sin\theta\,d\theta \int_0^{\frac{\pi}{2}} d\varphi \int_0^R f(\rho\cos\varphi\sin\theta,\ \rho\sin\varphi\sin\theta\ \rho\cos\theta)\,\rho^2\,d\rho.$

3550. $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{R\sqrt{\cos2\varphi}} \rho\,d\rho \int_{-\sqrt{R^2-\rho^2}}^{\sqrt{R^2-\rho^2}} f(\rho\cos\varphi,\ \rho\sin\varphi,\ z)\,dz.$

Rješenja

3551. $\displaystyle\int_0^{2\pi} d\varphi \int_0^{\frac{R\sqrt{3}}{2}} \rho\,d\rho \int_{R-\sqrt{R^2-\rho^2}}^{\sqrt{R^2-\rho^2}} f(\rho\cos\varphi,\ \rho\sin\varphi,\ z)\,dz$ ili

$$\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{3}} \sin\theta\,d\theta \int_0^R f(\rho\cos\varphi\sin\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\theta)\,\rho^2\,d\rho +$$

$$+\int_0^{2\pi} d\varphi \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin\theta\,d\theta \int_0^{2R\cos\theta} f(\rho\cos\varphi\sin\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\theta)\,\rho^2\,d\rho.$$

3552. $\frac{\pi a}{2}$. **3553.** $\frac{8}{9}a^2$. **3554.** $\frac{4}{15}\pi R^5$. **3555.** $\frac{\pi}{8}$.

3556. $\frac{4}{25}\pi(R^5-r^5)$. **3557.** $\frac{2\pi}{3}$.

3558. $\pi\left[3\sqrt{10}+\ln\frac{\sqrt{2}-1}{\sqrt{10}-3}-\sqrt{2}-8\right]$.

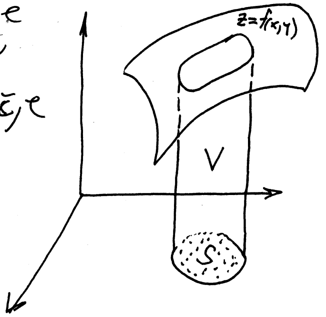
Primjena dvostrukog integrala

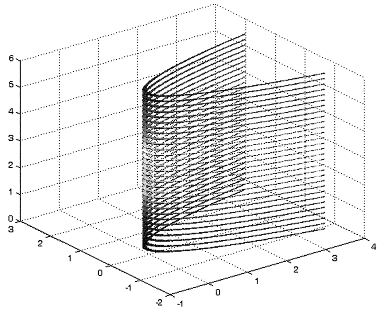
1° Površina zatvorene i ograničene oblasti D

$$P = \iint\limits_D dx\,dy$$

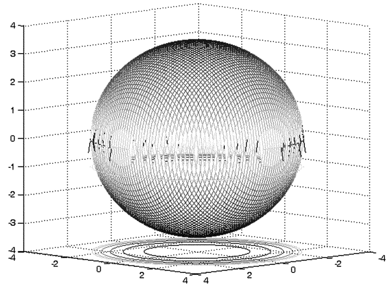
2° Zapremina tijela koje odozgo određuje površ $z=f(x,y)$, odozdo ravan $z=0$ a postranice valjkasta ploha koja na ravni XOY izrezuje omeđeno zatvoreno područje S iznosi

$$V = \iint\limits_S f(x,y)\,dx\,dy$$

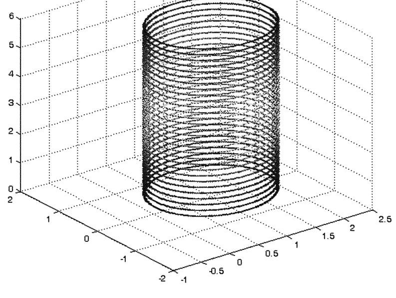




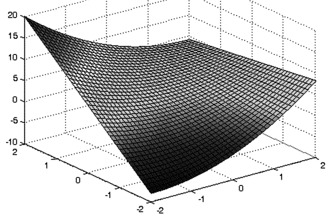
cilindar $x = 2y^2$



kugla $x^2 + y^2 + z^2 = 12$



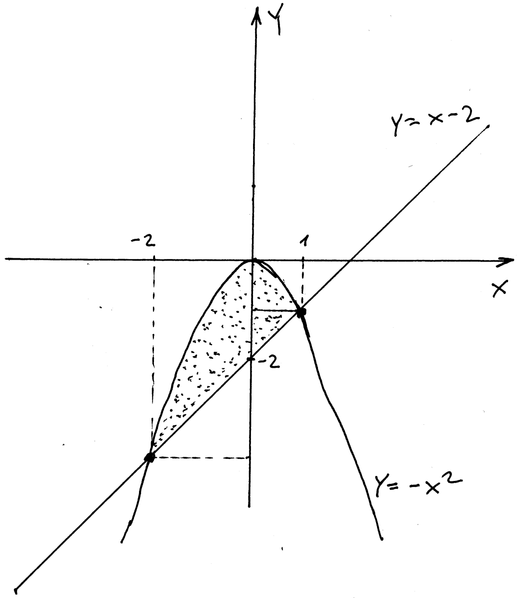
valjak $x^2 + y^2 = 2x$



funkcija $z = x^2 - 2xy + 3y + 2$

Ⓟ Naći površinu figure ograničene linijama $y=-x^2$, $x-y-2=0$.

Rj. Nacrtajmo sliku



Pronađimo presječne tačke krive $y=-x^2$ i prave $x-y-2=0$.

$$y=-x^2$$
$$x-y-2=0$$
$$x+x^2-2=0$$
$$x^2+x-2=0$$
$$D=1+8=9 \quad x_{1,2}=\frac{-1\pm3}{2}$$
$$x_1=-2,\quad x_2=1$$
$$(x-1)(x+2)=0$$
$$x=1 \Rightarrow y=-1$$
$$x=-2 \Rightarrow y=-4$$

I način:

$$P=\int_{-2}^{1}\left(-x^2-(x-2)\right)dx=\int_{-2}^{1}(-x^2-x+2)\,dx=\overbrace{-\frac{1}{3}x^3\Big|_{-2}^{1}}^{-1+8}\overbrace{-\frac{1}{2}x^2\Big|_{-2}^{1}}^{1-4}\overbrace{+2x\Big|_{-2}^{1}}^{1+2}=$$

$$=-\frac{1}{3}\cdot 9+\frac{1}{2}\cdot3+2\cdot3=-3+\frac{3}{2}+6=-3-\frac{3}{2}=\frac{9}{2}$$

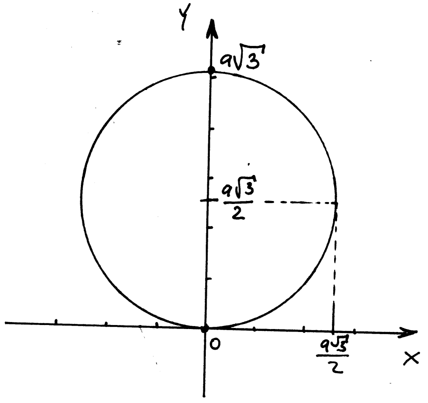
II način:

$$P=\iint\limits_D dx\,dy \quad \text{gdje je} \quad D:\begin{cases}-2\le x\le 1\\ x-2\le y\le -x^2\end{cases}$$

$$P=\iint\limits_D dx\,dy=\int_{-2}^{1}dx\int_{x-2}^{-x^2}dy=\int_{-2}^{1}\left((-x^2)-(x-2)\right)dx=\ldots=\frac{9}{2}$$

Ⓣ # Izračunati površinu figure koja je ograničena linijom $x^2+y^2=a\sqrt{3}\,y$.

Rj. $P=\iint\limits_D dx\,dy$



$$x^2+y^2=a\sqrt{3}\,y$$

$$x^2+y^2-a\sqrt{3}\,y=0$$

$$x^2+y^2-2\cdot\frac{a\sqrt{3}}{2}y+\frac{a^2\cdot 3}{4}-\frac{3a^2}{4}=0$$

$$x^2+\left(y-\frac{a\sqrt{3}}{2}\right)^2=\left(\frac{a\sqrt{3}}{2}\right)^2$$

krug s centrom u tački $C\left(0,\frac{a\sqrt{3}}{2}\right)$ poluprečnika $\frac{a\sqrt{3}}{2}$.

Uvodim smjene $x=r\cos\varphi$, $y=\frac{a\sqrt{3}}{2}+r\sin\varphi$ $\quad 0\le r\le\frac{a\sqrt{3}}{2}$, $0\le\varphi\le 2\pi$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi}\end{vmatrix}\qquad \frac{\partial x}{\partial r}=\cos\varphi\quad \frac{\partial x}{\partial\varphi}=-r\sin\varphi\qquad \frac{\partial y}{\partial r}=\sin\varphi\quad \frac{\partial y}{\partial\varphi}=r\cos\varphi$$

$dx\,dy=|J|\,dr\,d\varphi$

$J=r$

$$P=\iint\limits_D dx\,dy=\iint\limits_{D'}|r|\,dr\,d\varphi=\int_0^{2\pi}\left[\int_0^{\frac{a\sqrt{3}}{2}} r\,dr\right]d\varphi=$$

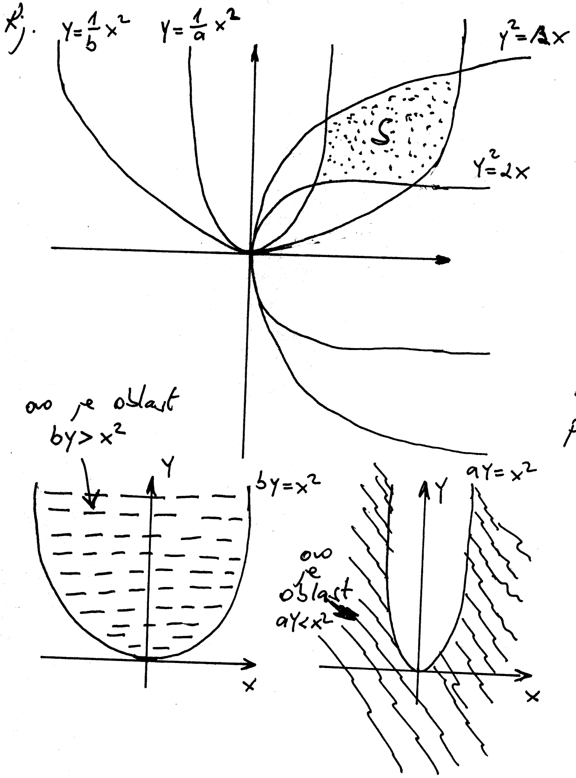
$$=\int_0^{2\pi}\frac{1}{2}r^2\Big|_0^{\frac{a\sqrt{3}}{2}}d\varphi=\frac{a^2\cdot 3}{4}\cdot\frac{1}{2}\varphi\Big|_0^{2\pi}=\frac{3a^2}{4}\cdot\pi$$

površina figure koja je ograničena linijom

Ⓣ # Izračunati površinu krivoliniskog 4-ugla omeđenog lukovima parabola $x^2=ay$, $x^2=by$, $y^2=\alpha x$ i $y^2=\beta x$ $(0<a<b,\ 0<\alpha<\beta)$.

$P=\iint\limits_S dx\,dy$

Rj.



$x^2=ay$ $\quad$ $x^2=by$

$y=\frac{1}{a}x^2$ $\quad$ $y=\frac{1}{b}x^2$

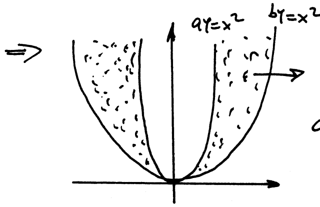
$a<b$

$\frac{1}{a}>\frac{1}{b}$ $\quad$ $\frac{1}{a}x^2>\frac{1}{b}x^2$

Na klasičan način površinu $\iint\limits_S dx\,dy$ je teško izračunati.

Primjetimo sljedeće:

$\Rightarrow$



oblast $ay\le x^2\le by$

$a\le\frac{x^2}{y}\le b$

Slično $y^2\ge\alpha x$ i $y^2\le\beta x$

imamo $\alpha x\le y^2\le\beta x$ $\quad$ $\alpha\le\frac{y^2}{x}\le\beta$

Vidimo da možemo uvesti smjene $u=\frac{x^2}{y}$ i $v=\frac{y^2}{x}$ gdje

$a\le u\le b$

$\alpha\le v\le\beta$

$$u=\frac{x^2}{y}\quad v=\frac{y^2}{x}$$

$$y=\frac{x^2}{u}.\quad x=\frac{y^2}{v}\quad\Rightarrow\quad x=\frac{\left(\frac{x^2}{u}\right)^2}{v}=\frac{x^4}{u^2v}\quad\Rightarrow\quad x^3=u^2v$$

$$x=\sqrt[3]{u^2v}$$

$$y=\frac{x^2}{u}=\frac{\sqrt[3]{(u^2v)^2}}{u}=\sqrt[3]{\frac{u^4v^2}{u^3}}=\sqrt[3]{uv^2}\qquad J=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\qquad dx\,dy=|J|\,du\,dv$$

$$x=u^{\frac{2}{3}}v^{\frac{1}{3}}\qquad \frac{\partial x}{\partial u}=\frac{2}{3}u^{-\frac{1}{3}}v^{\frac{1}{3}}\qquad \frac{\partial x}{\partial v}=u^{\frac{2}{3}}\frac{1}{3}v^{-\frac{2}{3}}$$

$$y=u^{\frac{1}{3}}v^{\frac{2}{3}}\qquad \frac{\partial y}{\partial u}=\frac{1}{3}u^{-\frac{2}{3}}v^{\frac{2}{3}}\qquad \frac{\partial y}{\partial v}=u^{\frac{1}{3}}\frac{2}{3}v^{-\frac{1}{3}}\qquad J=\frac{4}{9}-\frac{1}{9}=\frac{3}{9}=\frac{1}{3}$$

$$\iint\limits_S dx\,dy=\int_a^b\left[\int_\alpha^\beta\frac{1}{3}\,dv\right]du=\frac{1}{3}\int_a^b v\Big|_\alpha^\beta du=\frac{1}{3}(\beta-\alpha)u\Big|_a^b=\frac{1}{3}(b-a)(\beta-\alpha)$$

Ⓟ Izračunati zapreminu tijela, ograničeno površinama

$Y=x^2$, $Y=1$, $x+Y+z=4$, $z=0$.

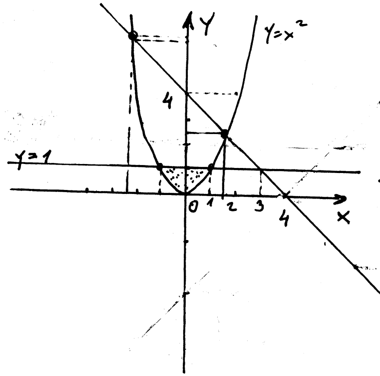
Rj. Skicirajmo naše tijelo.

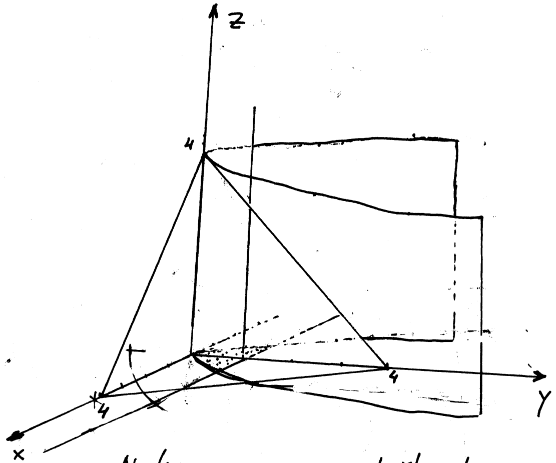
$x+Y+z=4$ je ravan $\left(\frac{x}{4}+\frac{Y}{4}+\frac{z}{4}=1\right)$ koja na x, Y, i z osi ima odsječke 4.

$Y=1$, $z=0$ su ravni

$Y=x^2$ je cilinda

Napravimo ortogonalne projekcije površina na xO_Y ravan





Nadimo presječnu tačku krive $Y=x^2$ i prave $x+Y=4$.

$$\frac{\begin{array}{l}Y=x^2\\ x+Y=4\end{array}}{\begin{array}{l}Y=x^2\\ Y=4-x\end{array}}$$

$x^2=4-x$

$x^2+x-4=0$

$D=1+16=17$

$x_{1,2}=\frac{-1\pm4{,}12}{2}$

$x_1\approx-\cancel{1}{,}56 \Downarrow Y_1=2{,}43$

$x_2\approx1{,}56 \Downarrow Y_2=6{,}56$

$V=\iint\limits_D f(x,Y)\,dx\,dY$ ← zapremina tijela koje je odozgo ograničeno sa i tijelo ima ortogonalnu projekciju D

U našem slučaju. $f(x,Y)=4-x-Y$ (ovo vidimo sa skice)

$V=\iint\limits_D(4-x-Y)\,dx\,dY$ gdje je $D:\begin{cases}-1\le x\le1\\ x^2\le Y\le1\end{cases}$. ili $D:\begin{cases}0\le Y\le1\\ -\sqrt{Y}\le x\le\sqrt{Y}\end{cases}$

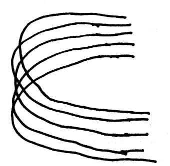
$$V=\int_{-1}^{1}dx\int_{x^2}^{1}(4-x-Y)\,dY=\int_{-1}^{1}\Big(\underset{4(1-x^2)}{4Y\Big|_{x^2}^{1}}-\underset{-x(1-x^2)}{xY\Big|_{x^2}^{1}}-\underset{-\frac{1}{2}(1-x^4)}{\frac{1}{2}Y^2\Big|_{x^2}^{1}}\Big)\,dx=$$

$$=\int_{-1}^{1}\left(4\underline{\underline{-4x^2}}\boxed{-x}\underline{+x^3}-\frac{1}{2}+\underline{\underline{\underline{\frac{1}{2}x^4}}}\right)dx=\int_{-1}^{1}\left(x^3-4x^2+\frac{1}{2}x^4-x+\frac{7}{2}\right)=\ldots=-\frac{8}{3}+\frac{1}{5}+7=\frac{68}{15}$$

traženo rješenje

Ⓟ Izračunati zapreminu tijela koje je ograničeno površinama $x=2Y^2$, $x+2Y+z=4$ i $z=0$.

Rj. $x=2Y^2$ cilindar u prostoru



Pronađimo projekciju površina na xO_Y ravan:

$$\frac{\begin{array}{l}x=2Y^2\\ x+2Y=4\end{array}}{2Y^2+2Y=4 \quad /:2}$$

$Y^2+Y-2=0$

$(Y-1)(Y+2)=0$

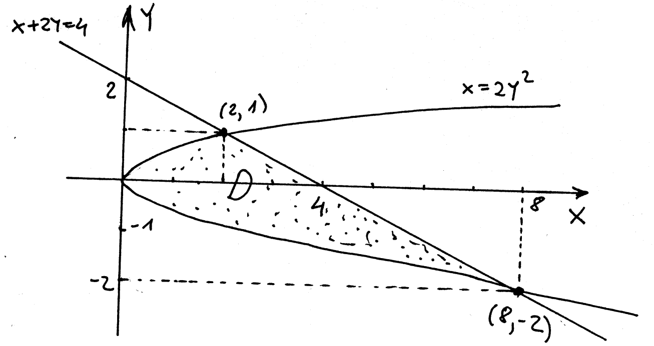
$Y_1=1 \Rightarrow x_1=2$

$Y_2=-2 \Rightarrow x_2=8$

$x+2Y=4 \quad /:4$

$\frac{x}{4}+\frac{Y}{2}=1$

Nacrtajmo sliku



$D:\begin{cases}-2\le Y\le1\\ 2Y^2\le x\le4-2Y\end{cases}$

$x+2Y+z=4$

$z=4-x-2Y$

$V=\iint\limits_D(4-x-2Y)\,dx\,dY$

$$V=\int_{-2}^{1}\left[\int_{2Y^2}^{4-2Y}(4-x-2Y)\,dx\right]dY=\int_{-2}^{1}\left[4x\Big|_{2Y^2}^{4-2Y}-\frac{1}{2}x^2\Big|_{2Y^2}^{4-2Y}-2Y\cdot x\Big|_{2Y^2}^{4-2Y}\right]dY=$$

$$=\int_{-2}^{1}\Big[\underset{16-8Y-8Y^2}{4(4-2Y-2Y^2)}-\frac{1}{2}\big(\underset{16-16Y+4Y^2-4Y^4}{(4-2Y)^2-(2Y^2)^2}\big)-2Y(4-2Y-2Y^2)\Big]dY$$

$$=\int_{-2}^{1}\left[16-\cancel{8Y}\underline{\underline{-8Y^2}}-8+\cancel{8Y}\underline{\underline{-2Y^2}}+\boxed{2Y^4}\underline{\underline{\underline{-8Y}}}\underline{\underline{+4Y^2}}\underline{\underline{\underline{+4Y^3}}}\right]dY=\int_{-2}^{1}(2Y^4-6Y^2+4Y^3-8Y+8)$$

$$=\frac{2}{5}Y^5\Big|_{-2}^{1}-\frac{6}{3}Y^3\Big|_{-2}^{1}+\frac{4}{4}Y^4\Big|_{-2}^{1}-\frac{8}{2}Y^2\Big|_{-2}^{1}+8Y\Big|_{-2}^{1}=\frac{2}{5}\cdot33-2\cdot9+1\cdot(-15)-\frac{8}{2}(-3)$$

$$+8\cdot3=\frac{66}{5}-18-15+12+24=\frac{66}{5}+36-33=\frac{66}{5}+\frac{15}{5}=\frac{81}{5}$$

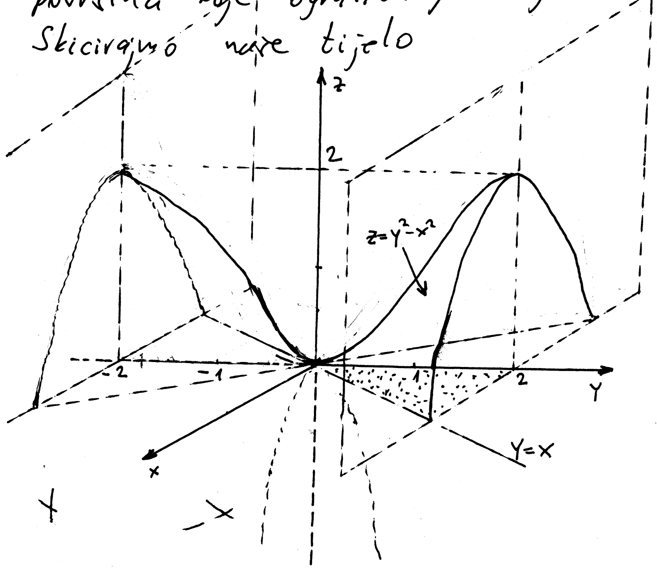
(#) Izračunati zapreminu tijela, koje je ograničeno sa površinama $z = y^2 - x^2$, $z = 0$, $y = \pm 2$.

Rj. Zapremina tijela se može računati pomoću dvostrukog ili pomoću trostrukog integrala. Za ta dva slučaja koristimo sljedeće dvije formule

$$V = \iint\limits_D f(x,y)\,dx\,dy \quad , \quad V = \iiint\limits_{\Omega} dx\,dy\,dz \, .$$

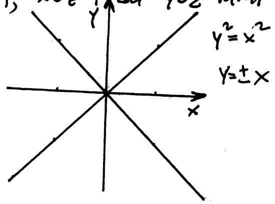
Koju od ove dvije formule je pogodnije koristiti zavisi od jednačina površina koje ograničavaju tijelo.

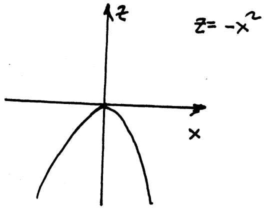
Skiciramo naše tijelo

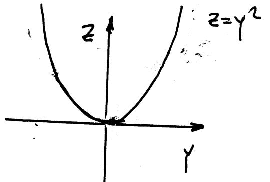


Šta predstavlja jednačina $z = y^2 - x^2$?

Napravimo presjeke $z=y^2-x^2$ sa xOy, xOz i sa yOz ravni







$$z(-x,-y) = (-y)^2 - (-x)^2 = y^2 - x^2 = z(x,y)$$

$\Rightarrow$ tijelo je simetrično u odnosu na koordinatni početak

$$z(x,-y) = (-y)^2 - x^2 = y^2 - x^2$$

$\Rightarrow$ tijelo je simetrično u odnosu na xOz osu

$z(-x,y) = y^2 - (-x)^2 = y^2 - x^2 \Rightarrow$ tijelo je simetričko u odnosu na YoZ -osu

Sa slike vidimo da možemo izabrati formulu za računanje zapremine $V = \iint\limits_D f(x,y)\,dx\,dy$ i to

$$V = 4\iint\limits_D (y^2 - x^2)\,dx\,dy \quad \text{gdje je} \quad D: \begin{cases} 0 \le y \le 2 \\ 0 \le x \le y \end{cases} \quad \text{(vidi sliku)}$$

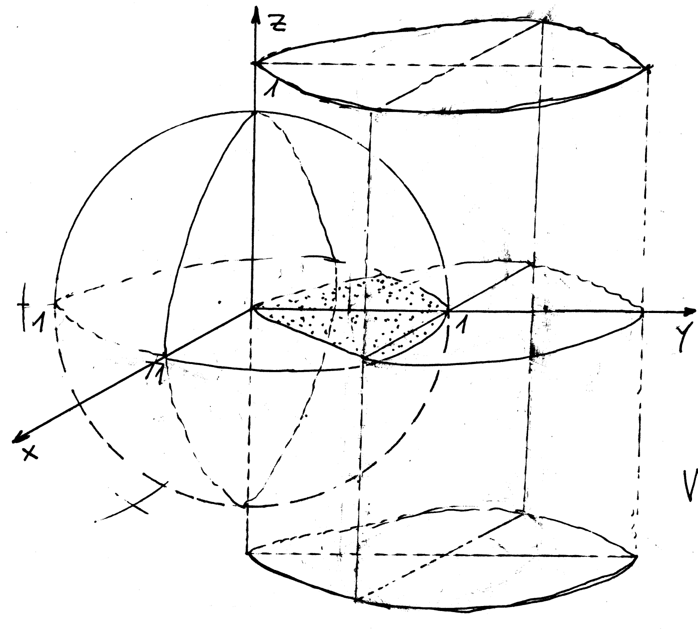
$$V = 4\int_0^2 dy \int_0^y (y^2 - x^2)\,dx = 4\int_0^2 \left(y^2 x \Big|_0^y - \frac{1}{3}x^3 \Big|_0^y \right) dy =$$

$$= 4\int_0^2 \left(y^3 - \frac{1}{3}y^3\right) dy = 4\int_0^2 \frac{2}{3}y^3\,dy = \frac{8}{3}\cdot\frac{1}{4}y^4\Big|_0^2 = \frac{8}{3}\cdot\frac{1}{\cancel{4}_1}\overset{4}{\cancel{16}} = \frac{32}{3}$$

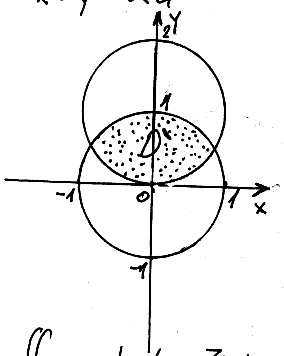
↑ traženo rješenje

\# Izračunati zapreminu onog dijela lopte $x^2+y^2+z^2=1$ koji se nalazi unutar cilindra $x^2+(y-1)^2=1$.

Rj. Nacrtajmo skicu ove dvije figure u prostoru



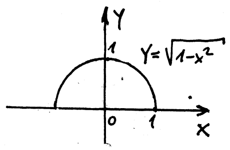
projekcije na xOy osu

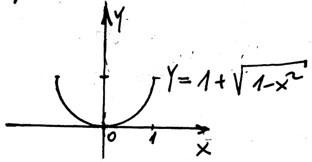


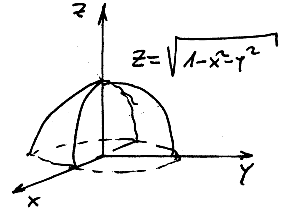
$$V=\iint_D z(x,y)\,dx\,dy$$

Zapremina tijela koje je odozgo ograničeno sa površi z a čija je projekcija na xOy ravan oblast D

Primjetimo da je presjek cilindra i lopte prvo simetričan u odnosu na xOy osu, a drugo da je simetričan u odnosu na yOz osu.







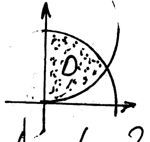
$$\frac{1}{4}V=\int_0^1 dx\int_{1+\sqrt{1-x^2}}^{\sqrt{1-x^2}}\sqrt{1-x^2-y^2}\,dy$$

$$\frac{1}{2}V=\iint_{D'} z(x,y)\,dx\,dy,\quad D':\begin{cases}-1\le x\le 1\\ 1+\sqrt{1-x^2}\le y\le\sqrt{1-x^2}\end{cases}$$

uvedimo polarne koordinate!

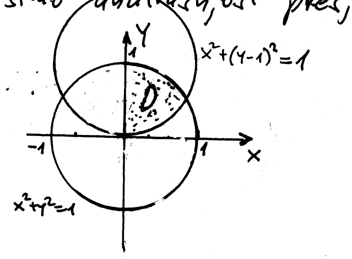
$x=r\cos\varphi$
$y=r\sin\varphi$
$dx\,dy=r\,dr\,d\varphi$

Kako opisati oblast



uz pomoć polarnih koordinata?

Opišimo unutrašnjost presjeka dva kruga pomoću polarnih koordinata.



$x^2+y^2\le 1$
$(r\cos\varphi)^2+(r\sin\varphi)^2\le 1$
$r^2\le 1$
$0\le r\le 1$

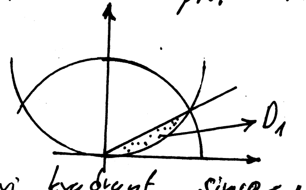
$x^2+(y-1)^2\le 1$
$x^2+y^2-2y+1\le 1$
$x^2+y^2\le 2y$
$r^2\le 2r\sin\varphi \quad /:r$
$r\le 2\sin\varphi$
$0\le r\le 2\sin\varphi$

Kako je $0\le\sin\varphi\le 1$ (ako posmatramo prvi kvadrant) to je moguć i slučaj da je $2\sin\varphi>1$ pa imamo dva slučaja

1° $2\sin\varphi\le 1 \Rightarrow \sin\varphi\le\frac{1}{2}$ (pa ako posmatramo prvi kvadrant) $(\sin\varphi\ge 0)$

$\Rightarrow \varphi\in(0,\frac{\pi}{6})$

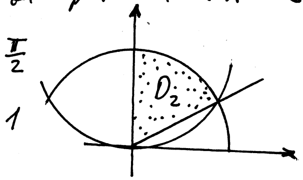
$$D_1:\begin{cases}0\le\varphi\le\frac{\pi}{6}\\ 0\le r\le 2\sin\varphi\end{cases}$$



2° $2\sin\varphi\ge 1 \Rightarrow \sin\varphi\ge\frac{1}{2}$ (pa za prvi kvadrant $\sin\varphi\le 1$)

$\Rightarrow \varphi\in(\frac{\pi}{6},\frac{\pi}{2})$

$$D_2:\begin{cases}\frac{\pi}{6}\le\varphi\le\frac{\pi}{2}\\ 0\le r\le 1\end{cases}$$



$D=D_1\cup D_2$

$$\frac{1}{4}V=\iint_D\sqrt{1-r^2}\,r\,dr\,d\varphi=\iint_{D_1}r\sqrt{1-r^2}\,dr\,d\varphi+\iint_{D_2}r\sqrt{1-r^2}\,dr\,d\varphi$$

$$\iint_{D_1}r\sqrt{1-r^2}\,dr\,d\varphi=\int_0^{\frac{\pi}{6}}d\varphi\int_0^{2\sin\varphi}r\sqrt{1-r^2}\,dr=\left|\begin{array}{l}d(1-r^2)=\\=-2r\,dr\end{array}\right|=\int_0^{\frac{\pi}{6}}d\varphi\int_0^{2\sin\varphi}\left(-\frac{1}{2}\right)\sqrt{1-r^2}\,d(1-r^2)$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{6}}\frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{2\sin\varphi}d\varphi=-\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\int_0^{\frac{\pi}{6}}\Big((1-\underbrace{4\sin^2\varphi}_{(2\sin\varphi)^2})^{\frac{3}{2}}-1\Big)\,d\varphi$$

Ovo je eliptički integral i on se ne mora izračunati. Njegova približna vrijednost je $\pi/18$.

$$\iint_{D_2}r\sqrt{1-r^2}\,dr\,d\varphi=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}d\varphi\int_0^1\left(-\frac{1}{2}\right)\sqrt{1-r^2}\,d(1-r^2)=\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}\left(-\frac{1}{2}\right)\frac{(1-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^1 d\varphi=-\frac{1}{3}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}}(-1)\,d\varphi$$

$$=\frac{1}{3}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\frac{1}{3}\cdot\frac{3\pi-\pi}{6}=\frac{1}{3}\cdot\frac{\overset{1}{2\pi}}{\underset{3}{\cancel{6}}}=\frac{\pi}{9}$$

$$\frac{1}{4}V=\frac{\pi}{9}+\frac{\pi}{18}=\frac{3\pi}{18}=\frac{\pi}{6}\qquad V=\frac{4\pi}{6}=\frac{2\pi}{3}$$

tražena zapremina

Izračunati zapreminu tijela ograničenog površima:

104. $z=x^2+y^2,\ y=x^2,\ x=1,\ z=0.$

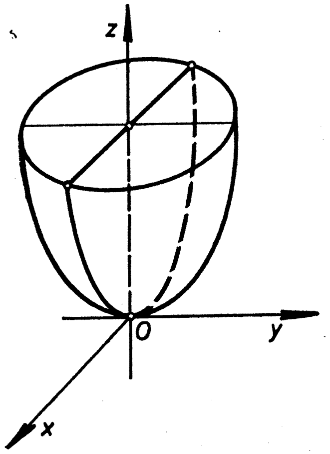
105. $z=xy,\ y=0,\ x=0,\ z=0,\ x^2+y^2=r^2.$

Rješenja:

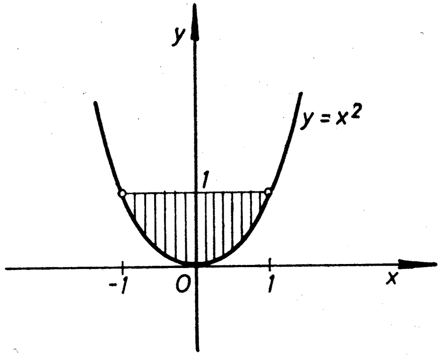
104. Zapremina tijela V ograničenog sa ravni $z=0$, površi $z=f(x, y)$ $(z\geqslant 0)$ i cilindrom koji izrezuje oblast D (x, y)-ravni, a ima izvodnice paralelne sa z-osom, data je sa

$$V=\iint_D f(x, y)\,dxdy.$$

U ovom slučaju površ $z=f(x, y)$ je paraboloid $z=x^2+y^2$, (slika 18a) dok je oblast D data na slici 18b.



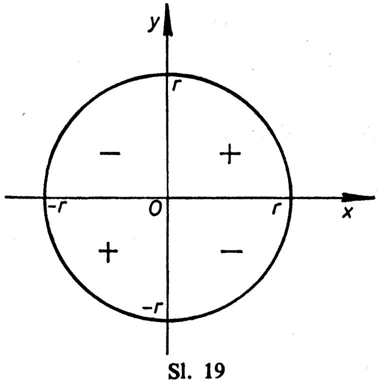
Sl. 18 a



Sl. 18 b

Biće

$$V=\iint_D (x^2+y^2)\,dxdy=\int_{-1}^{1} dx\int_{x^2}^{1}(x^2+y^2)\,dy=\frac{88}{105}.$$



Sl. 19

105. Tijelo V se sastoji iz četiri jednaka dijela od kojih su dva ispod ravni $z=0$ (sl. 19). Biće

$$V=4\int_0^r x\,dx\int_0^{\sqrt{r^2-x^2}} y\,dy=$$

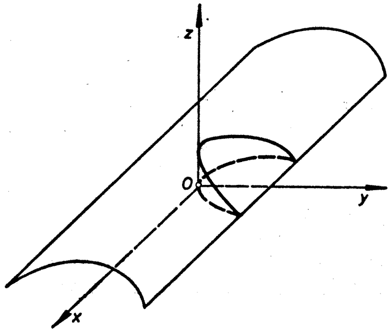
$$=2\int_0^r x\,(r^2-x^2)\,dx=\frac{r^4}{2}.$$

Izračunati zapreminu tijela ograničenog površima:

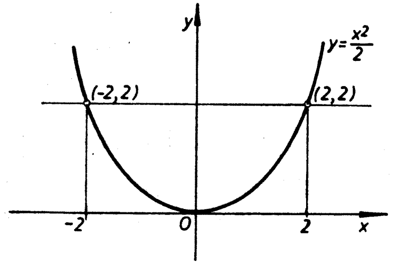
106. $z=4-y^2,\ y=\frac{x^3}{2},\ z=0.$

Rješenja:

106. Površ $z=4-y^2$ je parabolični cilindar okomit na ravan yOz, a površ $y=\frac{x^2}{2}$ je parabolični cilindar okomit na ravan xOy (sl. 20). Tijelo V projektuje se na oblast D u ravni $z=0$ ograničenu parabolom $y=\frac{x^2}{2}$ i presjekom cilindra $z=4-y^2$ i ravni $z=0$ (sl. 21).



Sl. 20



Sl. 21

$$V=\iint_D (4-y^2)\,dxdy=\int_{-2}^{2} dx\int_{\frac{x^2}{2}}^{2}(4-y^2)\,dy=2\int_0^2 dx\int_{\frac{x^2}{2}}^{2}(4-y^2)\,dy=$$

$$=2\int_0^2\left(4y-\frac{y^3}{3}\right)\Big|_{\frac{x^2}{2}}^{2} dx=2\int_0^2\left(8-\frac{8}{3}-2x^2+\frac{x^6}{24}\right)dx=\frac{256}{21}.$$

Izračunati zapreminu tijela ograničenog površima:

107. $z=1-4x^2-y^2,\ z=0.$

Rješenja:

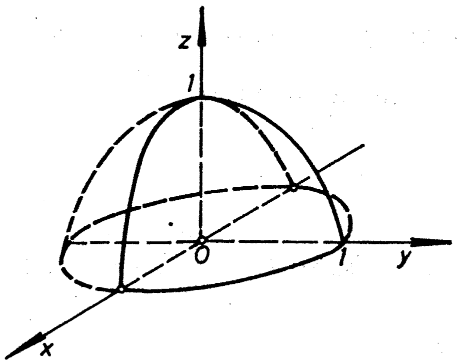
107. Paraboloid $z=1-4x^2-y^2$ je okrenut nadolje, i siječe se sa ravni $z=0$ po elipsi $4x^2+y^2=1$ (sl. 22 i sl. 23). Zato je

$$V=\iint\limits_D (1-4x^2-y^2)=\int\limits_{-1/2}^{1/2} dx \int\limits_{-\sqrt{1-4x^2}}^{\sqrt{1-4x^2}} (1-4x^2-y^2)\,dy=$$

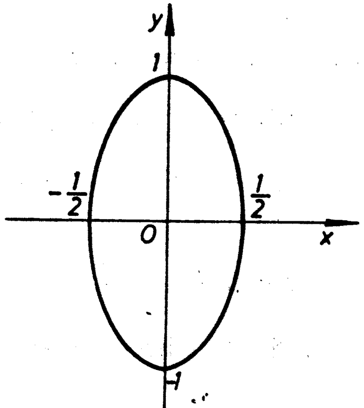
$$=4\int\limits_0^{1/2} dx \int\limits_0^{\sqrt{1-4x^2}} (1-4x^2-y^2)dy=\frac{8}{3}\int\limits_0^{1/2} (1-4x^2)^{\frac{3}{2}}\,dx.$$

Smjenom $2x=\sin t$ dobija se

$$V=\frac{4}{3}\int\limits_0^{\pi/2}\cos^4 t\,dt=\frac{4}{3}\int\limits_0^{\pi/2}\left(\frac{1+\cos 2t}{2}\right)^2 dt=\frac{\pi}{4}.$$



Sl. 22



Sl. 23

Zadaci za vježbu

Zapremina tela. I

U zadacima 3559 — 3596 pomoću dvojnih integrala naći zapremine tela ograničenih datim površima (parametre koji ulaze u uslove zadatka smatrati pozitivnim veličinama).

3559. Koordinatnim ravnima, ravnima $x=4$ i $y=4$ i obrtnim paraboloidom $z=x^2+y^2+1$.

3560. Koordinatnim ravnima, ravnima $x=a$, $y=b$ i eliptičnim paraboloidom $z=\frac{x^2}{2p}+\frac{y^2}{2q}$.

3561. Ravnima $x=0$, $y=0$, $z=0$ i $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ (piramida).

3562. Ravnima $y=0$, $z=0$, $3x+y=6$, $3x+2y=12$ i $x+y+z=6$.

3563. Obrtnim paraboloidom $z=x^2+y^2$, koordinatnim ravnima i ravni $x+y=1$.

3564. Obrtnim paraboloidom $z=x^2+y^2$ i ravnima $z=0$, $y=1$, $y=2x$ i $y=6-x$.

3565. Cilindrima $y=\sqrt{x}$, $y=2\sqrt{x}$ i ravnima $z=0$ i $x+z=6$.

3566. Cilindrom $z=\frac{1}{2}y^2$ i ravnima $x=0$, $y=0$, $z=0$ i $2x+3y-12=0$.

3567. Cilindrom $z=9-y^2$, koordinatnim ravnima i ravni $3x+4y=12$ $(y\geqslant 0)$.

3568. Cilindrom $z=4-x^2$, koordinatnim ravnima i ravni $2x+y=4$ $(x\geqslant 0)$.

3569. Cilindrom $2y^2=x$ i ravnima $\frac{x}{4}+\frac{y}{2}+\frac{z}{4}=1$ i $z=0$.

3570. Kružnim cilindrom poluprečnika r, čija se osa poklapa sa ordinatnom osom, koordinatnim ravnima i ravni $\frac{x}{r}+\frac{y}{a}=1$.

3571. Eliptičnim cilindorom $\frac{x^2}{4}+y^2=1$ i ravnima $z=12-3x-4y$ i $z=1$.

3572. Cilindrima $x^2+y^2=R^2$ i $x^2+z^2=R^2$.

3573. Cilindrima $z=4-y^2$, $y=\frac{x^2}{2}$ i ravni $z=0$.

3574. Cilindrima $x^2+y^2=R^2$, $z=\frac{x^3}{a^2}$ i ravni $z=0$ $(x\geqslant 0)$.

3575. Hiperboličnim paraboloidom $z=x^2-y^2$ i ravnima $z=0$ i $x=3$.

3576. Hiperboličnim paraboloidom $z=xy$, cilindrom $y=\sqrt{x}$ i ravnima $x+y=2$, $y=0$ i $z=0$.

3577. Paraboloidom $z=x^2+y^2$, cilindrom $y=x^2$ i ravnima $y=1$ i $z=0$.

3578. Eliptičnim cilindrom $\frac{x^2}{a^2}+\frac{z^2}{b^2}=1$ i ravnima $y=\frac{b}{a}x$, $y=0$ i $z=0$ $(x>0)$.

3579. Paraboloidom $z=\frac{a^2-x^2-4y^2}{a}$ i ravni $z=0$.

3580. Cilindrima $y=e^x$, $y=e^{-x}$, $z=e^2-y^2$ i ravni $z=0$.

3581. Cilindrima $y=\ln x$ i $z=\ln^2 x$ i ravnima $z=0$ i $y+z=1$.

3582*. Cilindrima $z=\ln x$ i $z=\ln y$ i ravnima $z=0$ i $x+y=2e$ $(x>1)$.

3583. Cilindrima $y=x+\sin x$, $y=x-\sin x$ i $z=\frac{(x+y)^2}{4}$ (parabolički cilindar čije su izvodnice paralelne pravoj $x-y=0$, $z=0$) i ravni $z=0$ $(0<x\leqslant<\pi,\ y>0)$.

Rješenja

3559. $186\frac{2}{3}$. **3560.** $\frac{ab}{6}\left(\frac{a^2}{p}+\frac{b^2}{q}\right)$.

3561. $\frac{abc}{6}$. **3562.** 12.

3563. $\frac{1}{6}$. **3564.** $78\frac{15}{32}$.

3565. $\frac{48}{5}\sqrt{6}$. **3566.** 16. **3567.** 45.

3568. $13\frac{1}{3}$. **3569.** $16\frac{1}{5}$.

3570. $ar^2\left(\frac{\pi}{4}-\frac{1}{3}\right)$. **3571.** 22π.

3572. $\frac{16}{3}R^3$. **3573.** $12\frac{4}{21}$.

3574. $\frac{4R^5}{15a^2}$. **3575.** 27. **3576.** $\frac{3}{8}$.

3577. $\frac{88}{105}$. **3578.** $\frac{1}{3}abc$.

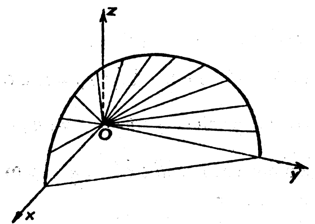
3579. $\frac{\pi a^3}{4}$. **3580.** $2\left(e^2-\frac{2e^3+1}{9}\right)$.

3581. $3e-8$.

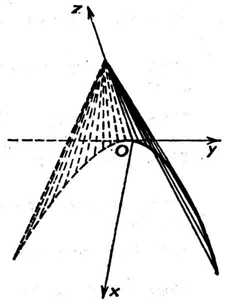
3582*. $4e-e^2-1$. Telo je simetrično u odnosu na ravan $y=x$.

3583. $2\left(\pi^2-\frac{35}{9}\right)$.

3584. Konusnom površinom $z^2=xy$ (sl. 66), cilindrom $\sqrt{x}+\sqrt{y}=1$ i ravni $z=0$.



Sl. 66



Sl. 67

3585. Konusnom površinom $4y^2=x(2-z)$ (parabolični konus, sl. 67) i ravnima $z=0$ i $x+z=2$.

3586. Površinom $z=\cos x\cdot\cos y$ i ravnima $x=0$, $y=0$, $z=0$ i $x+y=\frac{\pi}{2}$.

3587. Cilindrom $x^2+y^2=4$ i ravnima $z=0$ i $z=x+y+10$.

3588. Cilindrom $x^2+y^2=2x$ i ravnima $2x-z=0$ i $4x-z=0$.

3589. Cilindrom $x^2+y^2=R^2$, paraboloidom $Rz=2R^2+x^2+y^2$ i ravni $z=0$.

3590. Cilindrom $x^2+y^2=2ax$, paraboloidom $z=\frac{x^2+y^2}{a}$ i ravni $z=0$.

3591. Sferom $x^2+y^2+z^2=a^2$ i cilindrom $x^2+y^2=ax$ (Vivijanijev problem).

3592. Hiperboličkim paraboloidom $z=\frac{xy}{a}$, cilindrom $x^2+y^2=ax$ i ravni $z=0$ $(x>0,\ y>0)$.

3593. Cilindrima $x^2+y^2=x$ i $x^2+y^2=2x$, paraboloidom $z=x^2+y^2$ i ravnima $x+y=0$, $x-y=0$ i $z=0$.

3594. Cilindrima $x^2+y^2=2x$, $x^2+y^2=2y$ i ravnima $z=x+2y$ i $z=0$.

3595. Konusnom površinom $z^2=xy$ i cilindrom $(x^2+y^2)^2=2xy$ $(x>0,\ y>0,\ z\geqslant 0)$.

3596. Helikoidom („spiralne lestvice") $z=h\operatorname{arctg}\frac{y}{x}$, cilindrom $x^2+y^2=R^2$ i ravnima $x=0$ i $z=0$ $(x>0,\ y\geqslant 0)$.

Površina ravne oblasti

U zadacima 3597 — 3608 pomoću dvojnih integrala naći površine navedenih oblasti.

3597. Oblasti ograničene pravama $x=0$, $y=0$, $x+y=1$.

3598. Oblasti ograničene pravama $y=x$, $y=5x$, $x=1$.

3599. Oblasti ograničene elipsom $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

3600. Oblasti ograničene parabolom $y^2=\frac{b^2}{a}x$ i pravom $y=\frac{b}{a}x$.

3601. Oblasti ograničene parabolama $y=\sqrt{x}$, $y=2\sqrt{x}$ i pravom $x=4$.

3602*. Oblasti ograničene krivom $(x^2+y^2)^2=2ax^3$

3603. Oblasti ograničene krivom $(x^2+y^2)^3=x^4+y^4$.

3604. Oblasti ograničene krivom $(x^2+y^2)^2=2a^2(x^2-y^2)$ (Bernulijeva lemniskata).

3605. Oblasti ograničene petljom krive $x^3+y^3=2xy$ koja leži u prvom kvadrantu.

3606. Oblasti ograničene petljom krive $(x+y)^3=xy$ koja leži u prvom kvadrantu.

3607. Oblasti ograničene petljom krive $(x+y)^5=x^2y^2$ koja leži u prvom kvadrantu.

Rješenja

3584. $\frac{1}{45}$. **3585.** $\frac{16}{9}$. **3586.** $\frac{\pi}{4}$.

3587. 40π. **3588.** 2π.

3589. $\frac{5}{2}\pi R^3$. **3590.** $\frac{3}{2}\pi a^3$.

3591. $\frac{4}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)$. **3592.** $\frac{a^3}{24}$.

3593. $\frac{15}{8}\left(\frac{3\pi}{8}+1\right)$.

3594. $\frac{3}{2}\left(\frac{\pi}{2}-1\right)$. **3595.** $\frac{\pi\sqrt{2}}{24}$.

3596. $\frac{\pi^2R^2h}{16}$. **3597.** $\frac{1}{2}$.

3598. 2. **3599.** πab.

3600. $\frac{ab}{6}$. **3601.** $\frac{16}{3}$.

3602*. $\frac{5}{8}\pi a^2$; preći na polarne koordinate. **3603.** $\frac{3}{4}\pi$.

3604. $2a^2$. **3605.** $\frac{2}{3}$.

3606. $\frac{1}{60}$. **3607.** $\frac{1}{1260}$.

3608. Oblasti ograničene linijom

1) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{xy}{c^2}$; 2) $\left(\frac{x^2}{4}+\frac{y^2}{9}\right)^2=\frac{x^2+y^2}{25}$;

Površina površi

3626. Izračunati površinu onog dela ravni $6x+3y+2z=12$ koji leži u prvom oktantu.

3627. Izračuuati površinu onog dela površi $z^2=2xy$ koji se projektuje na pravougaonik u ravni $z=0$, ograničen pravama $x=0$. $y=0$, $x=3$, $y=6$.

3628. Naći površinu onog dela konusa $z^2=x^2+y^2$ koji leži između koordinatne ravni Oxy i ravni $z=\sqrt{2}\left(\frac{x}{2}+1\right)$.

U zadacima 3629 — 3639 naći površine nazn. a delova datih površi.

3629. Dela $z^2=x^2+y^2$ isečenog cilindrom $z^2=2py$.

3630*. Dela $y^2+z^2=x^2$, koji leži unatar cilindra $x^2+y^2=R^2$.

3631. Dela $y^2+z^2=x^2$, koji isecaju cilindar $x^2-y^2=a^2$ i ravni $y=b$ i $y=-b$.

3632. Dela $z^2=4x$, koji isecaju cilindar $y^2=4x$ i ravan $x=1$.

3633. Dela $z=xy$, isečenog cilindrom $x^2+y^2=R^2$.

3634. Dela $2z=x^2+y^2$, isečenog cilindrom $x^2+y^2=1$.

3635. Dela $x^2+y^2+z^2=a^2$, isečenog cilindrom $x^2+y^2=R^2$ $(R<a)$.

3636. Dela $x^2+y^2+z^2=R^2$, isečenog cilindrom $x^2+y^2=Rx$.

2637. Dela $x^2+y^2+z^2=R^2$, koga iseca „lemniskatni" cilindar $(x^2+y^2)^2=R^2(x^2-y^2)$.

3638. Dela $z=\frac{x+y}{x^2+y^2}$ koji leži u prvom oktantu i isečen je cilindrima $x^2+y^2=1$ i $x^2+y^2=4$.

3639. Dela $(x\cos\alpha+y\sin\alpha)^2+z^2=a^2$, koji leži u prvom oktantu $\left(\alpha<\frac{\pi}{2}\right)$.

3640*. Izračunati površinu dela zemljine kugle (smatrajući zemlju loptom poluprečnika $R\approx 6400\,km$) ograničenog meridijanima $\varphi=30°$ i $\varphi=60°$, i upodrednicima $\theta=45°$ i $\theta=60°$.

3641. Izračunati ukupnu površinu tela ograničenog sferom $x^2+y^2+z^2=3a^2$ i paraboloidom $x^2+y^2=2az$ $(z>0)$.

3642. Ose dva istovetna cilindra poluprečnika R seku se pod pravim uglom; naći površinu onog dela jednog cilindra koji leži u drugom cilindru.

Rješenja

3608*. 1) $\frac{a^2b^2}{2c^2}$; 2) $\frac{39}{25}\pi$; iskoristiti tvrđenje formulisano u zad. 3541.

3626. 14. **3627.** 36.

3628. 8π. **3269.** $2\sqrt{2}\pi p^2$

3630*. $2\pi R^2$. Projicirati površinu na ravan Oyz.

3631. $8\sqrt{2}ab$. **3632.** $\frac{16}{3}(\sqrt{8}-1)$.

3633. $\frac{2\pi}{3}\{(1+R^2)^{\frac{3}{2}}-1\}$.

3634. $\frac{2\pi}{3}(\sqrt{8}-1)$.

3635. $4\pi a(a-\sqrt{a^2-R^2})$.

3636. $2R^2(\pi-2)$.

3637. $2R^2(\pi+4-4\sqrt{2})$.

3638. $\frac{\pi}{4}\{3-\sqrt{2}-\sqrt{3}-\frac{\sqrt{2}}{2}\ln 2+\sqrt{2}\ln(\sqrt{3}+\sqrt{2})\}$.

3639. $\frac{2a^2}{\sin 2\alpha}$.

3640*. $\frac{\pi R^2}{12}(\sqrt{3}-\sqrt{2})\approx 3{,}42\cdot 10^8\,km^2$. Preći na sferne koordinate.

3641. $\frac{16}{3}\pi a^2$. **3642.** $8R^2$.

Primjena trostrukog integrala

a) Zapremina trodimenzionalnog tijela ograničenog oblašću Ω iznosi

$$V=\iiint\limits_{\Omega} dx\,dy\,dz$$

b) Težište $T(x_T, y_T, z_T)$ trodimenzionalnog homogenog tijela ograničenog oblašću Ω tražimo po formulama

$$x_T=\frac{1}{V}\iiint\limits_{\Omega} x\,dx\,dy\,dz, \qquad y_T=\frac{1}{V}\iiint\limits_{\Omega} y\,dx\,dy\,dz$$

$$z_T=\frac{1}{V}\iiint\limits_{\Omega} z\,dx\,dy\,dz$$

Homogeno tijelo je tijelo kojem je masa jednako raspoređena u svim njegovim dijelovima

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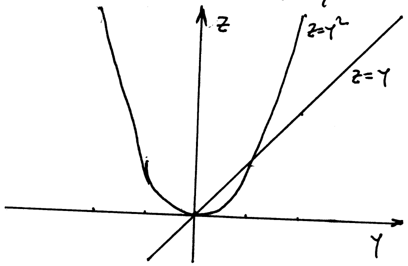
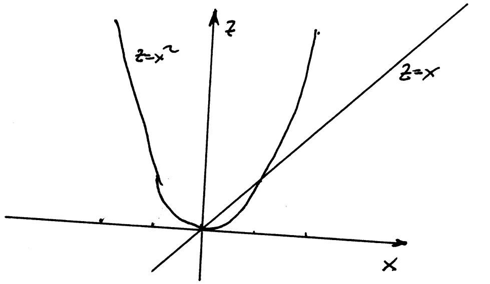
(#) Izračunati zapreminu tijela koju ravan $z=x+y$ odsijeca od paraboloida $z=x^2+y^2$.

Rj. Pogledajmo kako izgleda presjek datih površina sa yOz i xOz ravni





Na osnovu ove dvije slike pokušajte skicirati tijelo u prostoru!

$$V=\iiint\limits_{\Omega} dx\,dy\,dz=\iint\limits_{D} dx\,dy\int\limits_{x^2+y^2}^{x+y} dz=\iint\limits_{D}\left(x+y-(x^2+y^2)\right)dx\,dy \overset{(*)}{=}$$

gdje je D ortogonalna projekcija datog tijela na xOy ravan.

Projekciju presjeka tijela određujemo na sljedeći način

$$\begin{array}{l} z=x+y \\ z=x^2+y^2 \\ \hline \end{array}$$

$$x+y=x^2+y^2 \Rightarrow x^2-x+y^2-y=0$$

$$x^2-2\cdot x\cdot\frac{1}{2}+\frac{1}{4}+y^2-2\cdot y\cdot\frac{1}{2}+\frac{1}{4}=\frac{1}{2}$$

$$D:\ \left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{2}$$

Ako uvedemo polarne koordinate $x=\frac{1}{2}+r\cos\varphi$, $y=\frac{1}{2}+r\sin\varphi$, $dxdy=rdrd\varphi$

$$D \xrightarrow{\text{transformiše}} D'$$

$$D' : \begin{cases} 0 \leq r \leq \frac{1}{\sqrt{2}} \\ 0 \leq \varphi \leq 2\pi \end{cases}$$

$$\overset{(*)}{=} \iint_D (-1)(x^2 - x + y^2 - y)\, dx\, dy = (-1) \iint_D \left(\left(x - \frac{1}{2}\right)^2 - \left(y - \frac{1}{2}\right)^2 - \frac{1}{2} \right) dx\, dy =$$

Primjetimo da je $x - \frac{1}{2} = r\cos\varphi$

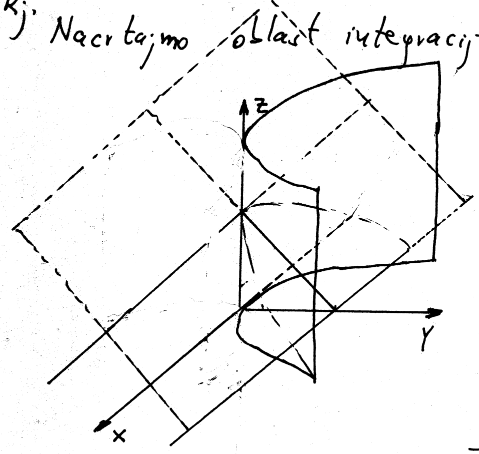
$y - \frac{1}{2} = r\sin\varphi$

$$= (-1) \iint_{D'} \left(r^2 - \frac{1}{2}\right) r\, dr\, d\varphi = (-1) \int_0^{2\pi} d\varphi \int_0^{1/\sqrt{2}} \left(r^3 - \frac{1}{2} r\right) dr = \ldots = \frac{\pi}{8}$$

traženo rješenje

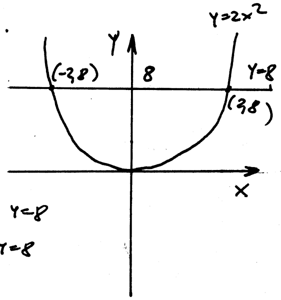
\# Izračunati zapreminu tijela koje je ograničeno cilindrom $y = 2x^2$ i ravnima $y + z = 8$, $z = 0$.

Rj. Nacrtajmo oblast integracije $\Omega : \begin{cases} y = 2x^2 \\ y + z = 8 \\ z = 0 \end{cases}$



Ravan $y + z = 8$ siječe cilindar

Napravimo projekciju oblasti Ω na xOy ravan.



Nađimo presjek krive $y = 2x^2$ i prave $y = 8$.

$$\begin{array}{l} y = 2x^2 \\ \underline{y = 8} \\ x^2 = 4 \\ x_1 = -2,\ x_2 = 2 \end{array}$$

$x_1 = -2 \Rightarrow y = 8$

$x_2 = 2 \Rightarrow y = 8$

$$\Omega : \begin{cases} -2 \leq x \leq 2 \\ 2x^2 \leq y \leq 8 \\ 0 \leq z \leq 8 - y \end{cases} \qquad V = \iiint_\Omega dx\, dy\, dz$$

$$V = \iiint_\Omega dx\, dy\, dz = \int_{-2}^{2} dx \int_{2x^2}^{8} dy \int_0^{8-y} dz = \int_{-2}^{2} dx \int_{2x^2}^{8} z \Big|_0^{8-y} dy = \int_{-2}^{2} dx \int_{2x^2}^{8} (8 - y)\, dy =$$

$$= \int_{-2}^{2} \left(8y \Big|_{2x^2}^{8} - \frac{1}{2} y^2 \Big|_{2x^2}^{8} \right) dx = \int_{-2}^{2} \left[8(8 - 2x^2) - \frac{1}{2}(8^2 - 4x^4) \right] dx =$$

$$= \int_{-2}^{2} (64 - 16x^2 - 32 + 2x^4)\, dx = \int_{-2}^{2} (2x^4 - 16x^2 + 32)\, dx =$$

$$= 2 \cdot \frac{1}{5} x^5 \Big|_{-2}^{2} - 16 \cdot \frac{1}{3} x^3 \Big|_{2}^{2} + 32x \Big|_{-2}^{2} = \frac{2}{5} \cdot 64 - \frac{16}{3} \cdot 16 + 32 \cdot 4 =$$

$$= \frac{384 - 1\,280 + 1\,920}{15} = \frac{1024}{15}$$

㊉ Izračunati zapreminu tijela ograničenog valjkom $x^2+y^2=6x$ i ravnima $x-z=0$, $5x-z=0$.

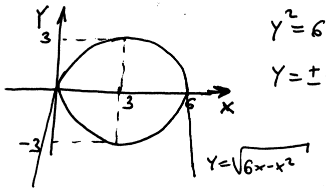
Rj. $V=\iiint_{\Omega} dx\,dy\,dz$

$x^2+y^2=6x$

$x^2-2\cdot x\cdot 3+3^2-3^2+y^2=0$

$(x-3)^2+y^2=3^2$

projekcija valjka na xOy ravan izgleda



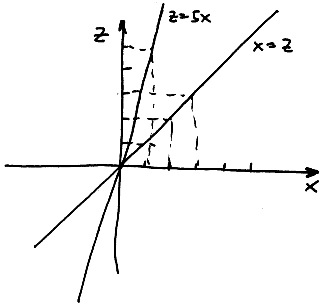
$Y^2=6x-x^2$

$Y=\pm\sqrt{6x-x^2}$

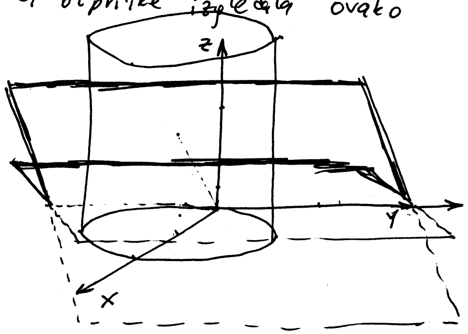
$x-z=0$
$x=z$

$5x-z=0$
$z=5x$

projekcije ravni $x-z=0$ i $5x-z=0$ na xOz ravan izgleda



Skica ovih figura u prostoru bi otprilike izgledala ovako



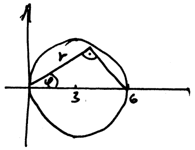
valjak presječen sa dvije ravni

na klasičan način

$$\Omega:\begin{cases}0<x<6\\ 0<y<\sqrt{6x-x^2}=\sqrt{9-(x-3)^2}\\ x\le z\le 5x\end{cases}$$

uvedimo cilindrične koordinate

$x=r\cos\varphi$
$Y=r\sin\varphi$
$z=z$
$dx\,dy=r\,dr\,d\varphi$



$\cos\varphi=\frac{r}{6}$

Primjetimo da je oblast Ω simetrična u odnosu na xOz ravan

$V=2\iiint_{\Omega'} r\,dr\,d\varphi\,dz$

$$\Omega':\begin{cases}0\le\varphi\le\frac{\pi}{2}\\ 0\le r\le 6\cos\varphi\\ r\cos\varphi<z<5r\cos\varphi\end{cases}$$

$$V=2\iiint_{\Omega'} r\,dr\,d\varphi\,dz=2\int_0^{\pi/2}d\varphi\int_0^{6\cos\varphi}r\,dr\underbrace{\int_{r\cos\varphi}^{5r\cos\varphi}dz}_{=4r\cos\varphi}=8\int_0^{\pi/2}\cos\varphi\,d\varphi\int_0^{6\cos\varphi}r^2dr=8\int_0^{\pi/2}\frac{1}{3}r^3\Big|_0^{6\cos\varphi}\cos\varphi\,d\varphi$$

$$=8\cdot 6^3/3\int_0^{\pi/2}\cos^4\varphi\,d\varphi=576\int_0^{\pi/2}\left(\frac{1}{2}(1+\cos2\varphi)\right)^2d\varphi=144\int_0^{\pi/2}(1+2\cos2\varphi+\cos^2 2\varphi)d\varphi=\overset{\text{za vježbu}}{\ldots}=108\pi$$

tražena zapremina ↗

㊉ Izračunati zapreminu tijela ograničenog ravninom XoY, valjkom $x^2+y^2=2ax$ i čunjem $x^2+y^2=z^2$.

Rj. Zapremina trodimenzionalnog tijela ograničenog oblašću Ω iznosi $V=\iiint_{\Omega} dx\,dy\,dz$. Pokušajmo skicirati tijelo čiju zapreminu tražimo.

valjak $x^2+y^2=2ax$

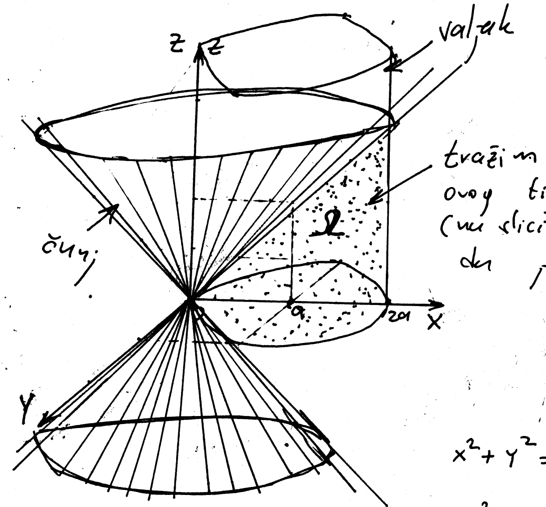
$x^2-2ax+y^2=0$

$x^2-2\cdot x\cdot a+a^2-a^2+y^2=0$

$(x-a)^2+y^2=a^2$

valjak u presjeku sa XoY ravni je krug sa centrom u tački $(a,0)$ poluprečnika a

čunj $x^2+y^2=z^2$ u presjeku sa XoY ravni je tačka, a u presjeku sa YoZ ili sa XoZ su po dvije prave



tražimo zapreminu ovog tijela (na slici smo pretpostavili da je $a>0$)

Oblast Ω je najlakše projicirati na XoY ravan.

Uvodimo cilindrične koordinate

$x=a+r\cos\varphi$
$Y=r\sin\varphi$
$z=z$

$$\Omega':\begin{cases}dx\,dy\,dz=r\,dr\,d\varphi\,dz\\ 0\le r\le a\\ 0\le\varphi\le 2\pi\end{cases}$$

$x^2+y^2=z^2$

$z=\pm\sqrt{x^2+y^2}$ čunj

$x^2+y^2=(a+r\cos\varphi)^2+(r\sin\varphi)^2=$

$=a^2+2ar\cos\varphi+r^2\cos^2\varphi+r^2\sin^2\varphi=$

$=a^2+2ar\cos\varphi+r^2$

$$V=\iiint_{\Omega}dx\,dy\,dz=\iiint_{\Omega'}r\,dr\,d\varphi\,dz=\int_0^a dr\int_0^{2\pi}d\varphi\int_0^{\sqrt{a^2+2ar\cos\varphi+r^2}}r\,dz=\ldots$$

ovo je teško izračunati

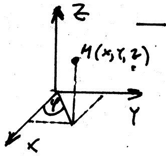
Pokušajmo uvesti drugačije smjene.

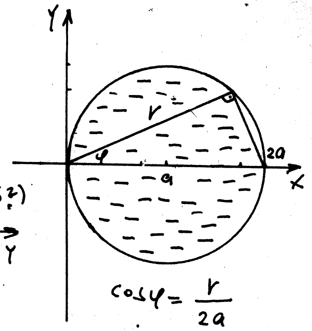
$x = r\cos\varphi$ $\quad dx\,dy\,dz = r\,dr\,d\varphi\,dz$

$y = r\sin\varphi$

$z = z$ $\quad x^2+y^2 = r^2\cos^2\varphi + r^2\sin^2\varphi = r^2$

$$\Omega'': \begin{cases} -\pi/2 \le \varphi \le \pi/2 \\ 0 \le r \le 2a\cos\varphi \\ 0 \le z \le \sqrt{r^2} \end{cases}$$





$\cos\varphi = \dfrac{r}{2a}$

$r = 2a\cos\varphi$

$$V = \iiint_{\Omega} dx\,dy\,dz = \iiint_{\Omega''} r\,dr\,d\varphi\,dz =$$

$$= \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} dr \int_0^{r} r\,dz = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} \left(rz\Big|_0^r \right) dr = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^2\,dr =$$

$$= \int_{-\pi/2}^{\pi/2} \left(\frac{1}{3} r^3 \Big|_0^{2a\cos\varphi} \right) d\varphi = \int_{-\pi/2}^{\pi/2} \frac{8}{3} a^3 \cos^3\varphi\,d\varphi = \frac{8}{3} a^3 \int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi$$

$$\int_{-\pi/2}^{\pi/2} \cos^3\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi\cos^2\varphi\,d\varphi = \int_{-\pi/2}^{\pi/2} \cos\varphi(1-\sin^2\varphi)\,d\varphi = \left| \begin{array}{l} \sin\varphi = t \\ \cos\varphi\,d\varphi = dt \\ \varphi = -\frac{\pi}{2} \Rightarrow t = -1 \\ \varphi = \frac{\pi}{2} \Rightarrow t = 1 \end{array} \right|$$

$$= \int_{-1}^{1} (1-t^2)\,dt = t\Big|_{-1}^{1} - \frac{1}{3} t^3\Big|_{-1}^{1} = 2 - \frac{1}{3}\cdot 2 = \frac{4}{3}$$

$V = \dfrac{32}{9} a^3$ tražena zapremina

II način: $V = \iint_S f(x,y)\,dx\,dy$ uvedimo smjene $\quad x = r\cos\varphi$

$y = r\sin\varphi$

$0 \le \varphi \le 2a\cos\varphi$

$-\frac{\pi}{2} \le r \le \frac{\pi}{2}$

$$V = \int_{-\pi/2}^{\pi/2} d\varphi \int_0^{2a\cos\varphi} r^2\,dr$$

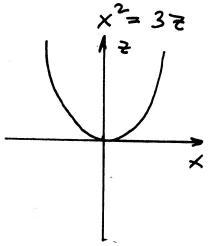
ZAVRŠITI ZA VJEŽBU

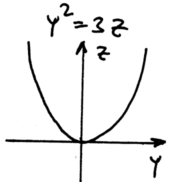
(#) Izračunati zapreminu tijela koje je ograničeno površima $x^2+y^2+z^2=4$ i $x^2+y^2=3z$.

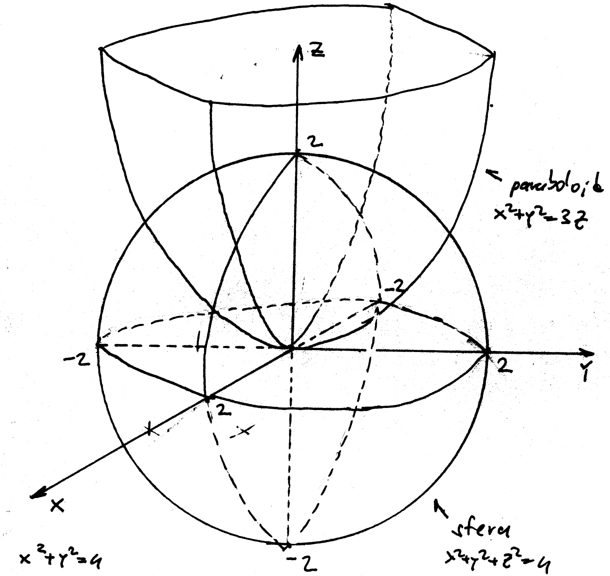
Rj. $x^2+y^2+z^2=4$ je sfera sa centrom u (0,0,0) poluprečnika 2

$x^2+y^2=3z$ je paraboloid

Skicirajmo ova dva tijela







$x^2+y^2=4$

$x = \pm\sqrt{4-y^2}$

$y = \pm\sqrt{4-x^2}$

$$V = \iiint_{\Omega} dx\,dy\,dz$$

Primetimo da je tijelo dobijeno presjekom simetrično na ravni xOz i na yOz.

Prema tome

$V = 4\iiint_{\Omega_1} dx\,dy\,dz$ gdje je

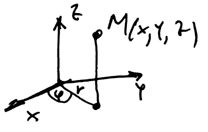
Ω_1 oblast u presjeku dva tijela u prvom oktantu

$$\Omega_1: \begin{cases} 0 \le x \le 2 \\ 0 \le y \le \sqrt{4-x^2} \\ 0 \le z \le \frac{1}{3}(x^2+y^2) \end{cases}$$

$$V = 4\int_0^2 dx \int_0^{\sqrt{4-x^2}} dy \int_0^{\frac{1}{3}(x^2+y^2)} dz = 4\int_0^2 dx \int_0^{\sqrt{4-x^2}} \frac{1}{3}(x^2-y^2)\,dy$$

$$= \frac{4}{3}\int_0^2 \left(x^2 y\Big|_0^{\sqrt{4-x^2}} + \frac{1}{3} y^3\Big|_0^{\sqrt{4-x^2}} \right) dx = \underset{\nearrow \ldots}{=} \frac{8\pi}{3}$$

komplikovano



II način:

Uvedimo cilindrične koordinate $\quad x = r\cos\varphi$

$y = r\sin\varphi$

$z = z$

$dx\,dy\,dz = r\,dr\,d\varphi\,dz$

Oblast $\Omega_1 \xrightarrow{\text{transformiše}} \Omega_1' : \begin{cases} 0 \le r \le 2 \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le z \le \frac{1}{3} r^2 \end{cases}$

$$V = 4\int_0^{\frac{\pi}{2}} d\varphi \int_0^2 dr \int_0^{\frac{1}{3}r^2} r\,dz = 4\int_0^{\frac{\pi}{2}} d\varphi \int_0^2 r \cdot \frac{1}{3} r^2\,dr = \frac{4}{3}\int_0^{\frac{\pi}{2}} \frac{1}{4} r^4 \Big|_0^2 d\varphi = \frac{1}{3} \cdot 16 \cdot \frac{\pi}{2} = \frac{8\pi}{3}$$

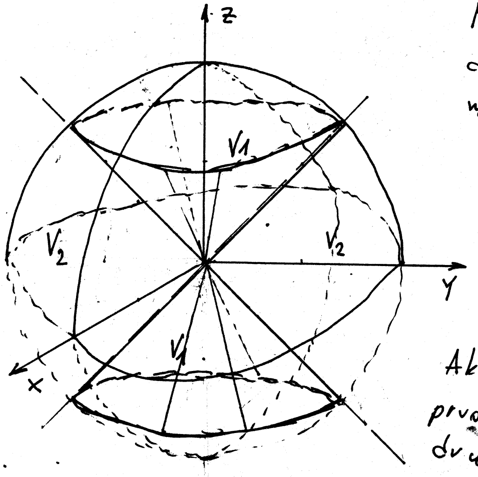
$V = \frac{8\pi}{3}$ tražena zapremina

Izračunati zapreminu tijela koje je ograničeno površima $z^2 = x^2 + y^2$, $x^2 + y^2 + z^2 = 4$.

Rj. $x^2+y^2+z^2=4$ je kugla sa centrom u $(0,0,0)$ poluprečnika $r=2$

$z^2 = x^2 + y^2$ je konus

Skicirajmo ove dvije figure u prostoru.



Presjek konusa i kugle daje dva tijela za koje možemo računati zapreminu: prvo tijelo je određeno u presjeku unutrašnjosti konusa i kugle, a drugo tijelo je određeno dijelom lopte van konusa. Ako sa V_1 označimo zapreminu prvog, a sa V_2 zapreminu drugog tijela, imamo da je

$V = V_1 + V_2 = \frac{4}{3} r^3 \pi$ (zapremina kugle)

Kako je $r=2 \Rightarrow V = \frac{4}{3} \cdot 8\pi = \frac{32\pi}{3}$.

$V = \iiint_\Omega dx\,dy\,dz$ – zapremina tijela ograničenog sa oblašću Ω

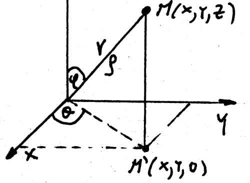
Uvedimo sferne koordinate

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

$z = \rho \cos\varphi$

$dx\,dy\,dz = \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$



$z^2 = x^2 + y^2$

$\rho^2 \cos^2\varphi = \rho^2 \sin^2\varphi \cos^2\theta + \rho^2 \sin^2\varphi \sin^2\theta =$

$= \rho^2 \sin^2\varphi (\cos^2\theta + \sin^2\theta) = \rho^2 \sin^2\varphi$

$\Rightarrow \cos^2\varphi = \sin^2\varphi \quad /: \sin^2\varphi$

$tg^2\varphi = 1 \Rightarrow tg\varphi = \pm 1$

$x^2+y^2+z^2=4 \Rightarrow \rho^2 = 4 \Rightarrow \rho = \pm 2$ tj. $\rho = 2$ udaljenost tačke

$\Omega : \begin{cases} z^2 = x^2 + y^2 \\ x^2 + y^2 + z^2 = 4 \end{cases} \xrightarrow{\text{transformiše}} \Omega' : \begin{cases} tg\varphi = \pm 1 \\ \rho = 2 \end{cases}$

Odredimo granice za drugo tijelo $\Omega'_{V_2}: \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ \frac{\pi}{4} \le \varphi \le \frac{3\pi}{4} \end{cases}$

$$V_2 = \iiint\limits_{\Omega'_{V_2}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho = \int_0^{2\pi} d\theta \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\varphi \, d\varphi \int_0^2 \rho^2 d\rho =$$

$$= 2\pi \cdot (-\cos\varphi)\Big|_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \frac{1}{3}\rho^3\Big|_0^2 = 2\pi\left(-\cos\frac{3\pi}{4} + \cos\frac{\pi}{4}\right) \cdot \frac{8}{3} =$$

$$= 2\pi\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3} = 2\pi\sqrt{2} \cdot \frac{8}{3} = \frac{16\pi\sqrt{2}}{3}$$ traženo rješenje

Zapreminu V_1 sad možemo odrediti na dva načina

I način:

$$V = V_1 + V_2 = \frac{32\pi}{3} \Rightarrow V_1 = \frac{32\pi}{3} - V_2 = \frac{32\pi}{3} - \frac{16\pi\sqrt{2}}{3}$$

$$V_1 = \frac{16\pi}{3}(2 - \sqrt{2})$$ traženo rješenje

II način:

Ako uzmemo u obzir simetričnost date oblasti Ω' u odnosu na xOy-ravan, možemo računati polovinu zapremine V_1 za $z \geq 0$ i tada bi trebalo odabrati sljedeće granice $\Omega'_{V_1}: \begin{cases} 0 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \varphi \le \frac{\pi}{4} \end{cases}$ $\quad V_1 = \iiint\limits_{\Omega'_{V_1}} \rho^2 \sin\varphi \, d\theta \, d\varphi \, d\rho$

$$\frac{1}{2}V_1 = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\varphi \, d\varphi \int_0^2 \rho^2 d\rho = 2\pi(-\cos\varphi)\Big|_0^{\pi/4} \cdot \frac{\rho^3}{3}\Big|_0^2 =$$

$$= 2\pi\left(1 - \cos\frac{\pi}{4}\right) \cdot \frac{8}{3} = 2\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3}$$

$$\Rightarrow V_1 = 4\pi\left(1 - \frac{\sqrt{2}}{2}\right) \cdot \frac{8}{3} = 4\pi \cdot \frac{2 - \sqrt{2}}{2} \cdot \frac{8}{3} = \frac{16\pi}{3}(2 - \sqrt{2}).$$

(#) Izračunati zapreminu tijela koje je određeno oblašću $\Omega: |x+y+z| + |x-y+z| + |x+y-z| = 1$.

Rj. $V = \iiint\limits_{\Omega} dx\,dy\,dz$

uvedimo smjenu $\quad u = x+y+z \quad v = x-y+z \quad w = x+y-z \qquad dx\,dy\,dz = J\,du\,dv\,dw$

$\uparrow$ Jakobijan

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \qquad J^{-1} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \begin{matrix} \underline{\underline{I_v + III_v}} \\ II_v + III_v \end{matrix}$$

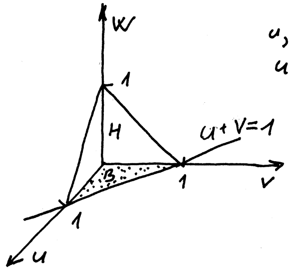
$$= \begin{vmatrix} 2 & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -1 \end{vmatrix} = (-1) \cdot \begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = (-1)(-4) = 4 \Rightarrow$$

pa je

$dx\,dy\,dz = \frac{1}{4}du\,dv\,dw$ $\qquad \Rightarrow J = \frac{1}{4}$

$\Omega': |u| + |v| + |w| = 1$

$$V = \iiint\limits_{\Omega'} \frac{1}{4} du\,dv\,dw$$



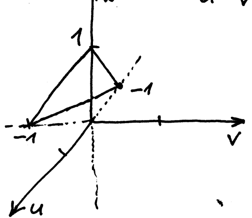
$u, v, w > 0$

$u + v + w = 1$

pored ovoga imamo još 7 slučajeva

npr. $u, v < 0 \quad w > 0$

$-u - v + w = 1$



Vidimo da je dovoljno oblast integrirati u 1. oktantu jer imamo simetričnu oblast po svim oktantima.

$$V = 8 \cdot \frac{1}{4} \iiint\limits_{\Omega''} du\,dv\,dw =$$

$$= 2\int_0^1 du \int_0^{1-u} dv \int_0^{1-u-v} dw = 2\int_0^1 du \int_0^{1-u} w\Big|_0^{1-u-v} dv =$$

$u + v = 1$

$v = 1 - u$

$u + v + w = 1$

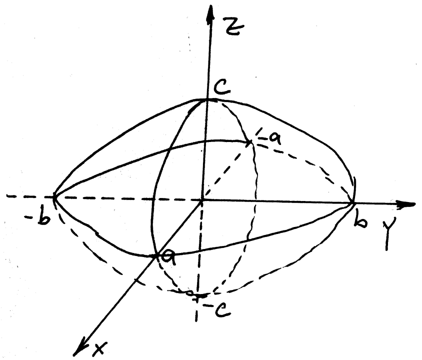
$w = 1 - u - v$

$$= 2\int_0^1 du \int_0^{1-u} (1 - u - v)\,dv = 2\int_0^1 \left(v\Big|_0^{1-u} - uv\Big|_0^{1-u} - \frac{1}{2}v^2\Big|_0^{1-u}\right) du = \ldots = 2 \cdot \frac{1}{6} = \frac{1}{3}$$

Na drugi način: $V_1 = \frac{B \cdot H}{3} = \frac{\frac{1 \cdot 1}{2} \cdot H}{3} = \frac{\frac{1}{2} \cdot 1}{3} = \frac{1}{6}, \quad V = 2 \cdot \frac{1}{6} = \frac{1}{3}$ zapremina tijela

(#) Izračunati zapreminu elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Rj:



$$V=\iiint_S dx\,dy\,dz$$

$$S:\ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

smjena: uopštene sferne koordinate

$x=ar\sin\varphi\cos\theta$ $\quad 0\le r\le 1$

$y=br\sin\varphi\sin\theta$ $\quad 0\le\varphi\le\pi$

$z=cr\cos\varphi$ $\quad 0\le\theta\le 2\pi$

$$dx\,dy\,dz=J\,dr\,d\varphi\,d\theta$$

$$J=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\varphi} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\varphi} & \frac{\partial y}{\partial\theta}\\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\varphi} & \frac{\partial z}{\partial\theta}\end{vmatrix}=\begin{vmatrix}a\sin\varphi\cos\theta & ar\cos\varphi\cos\theta & -ar\sin\varphi\sin\theta\\ b\sin\varphi\sin\theta & br\cos\varphi\sin\theta & br\sin\varphi\cos\theta\\ c\cos\varphi & -cr\sin\varphi & 0\end{vmatrix}$$

$$=abc\begin{vmatrix}\text{ista determinanta}\\ \text{kao kod standardnih}\\ \text{sfernih koordinata}\end{vmatrix}=abcr^2\sin\varphi$$

$$V=\int_0^{\pi}d\varphi\int_0^{1}dr\int_0^{2\pi}abcr^2\sin\varphi\,d\theta=\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{1}r^2\,dr\int_0^{2\pi}abc\,d\theta=$$

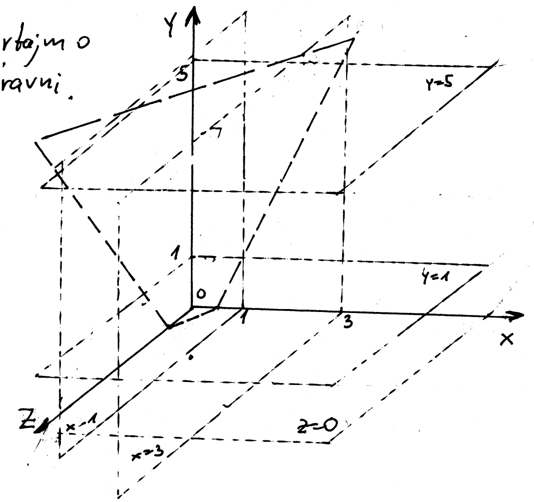
$$=abc\,\theta\Big|_0^{2\pi}\int_0^{\pi}\sin\varphi\,d\varphi\int_0^{1}r^2dr=2\pi abc\int_0^{\pi}\sin\varphi\,\frac{1}{3}r^3\Big|_0^1d\varphi=$$

$$=\frac{2}{3}\pi abc\int_0^{\pi}\sin\varphi\,dy=\frac{2}{3}\pi abc\left(-\cos\varphi\Big|_0^{\pi}\right)=\frac{2}{3}\pi abc(1+1)=\frac{4}{3}\pi abc$$

q.e.d.

(#) Naći zapreminu tijela ograničenog ravnima $x=1$, $x=3$, $y=1$, $y=5$, $2x-y+z-1=0$, $z=0$.

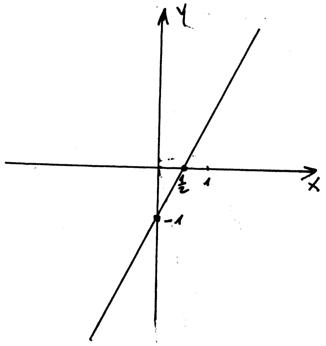
Rj. Nacrtajmo ove ravni.



$2x-y+z-1=0$

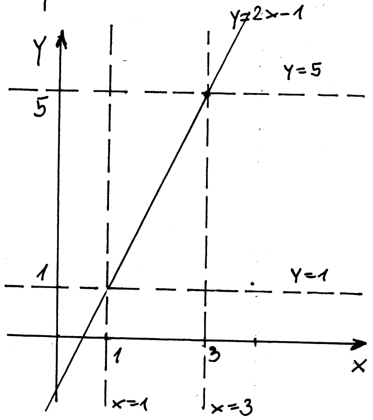
$z=-2x+y+1$

projekcija ove ravni na xOy ravan



$y=2x-1$

Slika u prostoru je komplikovana i sa nje ne možemo pročitati granice. Nacrtajmo projekcije ovih ravni na xOy ravan.



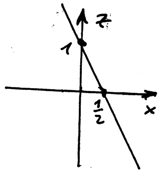
$2x-y-1=0$

$y=2x-1$

$x=3\Rightarrow y=5$

$x=1\Rightarrow y=1$

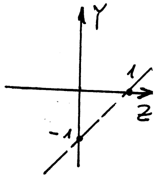
na xOz ravan



$2x+z-1=0$

$z=-2x+1$

na yOz ravan



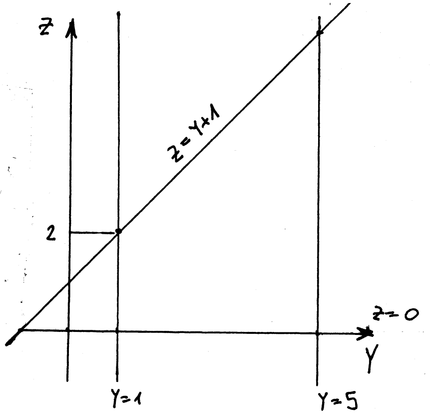
$-y+z-1=0$

$y=z-1$

Sad na osnovu slike u prostoru i projekcija na ravni možemo pročitati granice za tijelo

$$\Omega:\begin{cases}1\le x\le 3\\ 2x-1\le y\le 5\\ 0\le z\le -2x+y+1\end{cases}$$

Da su napisane granice ispravne provjerimo projekcijom ravni na yOz ravan.



$$-y+z-1=0$$
$$z=y+1$$

$$V=\iiint\limits_{\Omega} dx\,dy\,dz =$$

$$=\int_1^3 dx\int_{2x-1}^{5} dy \underbrace{\int_0^{-2x+y+1} dz}_{z\Big|_0^{-2x+y+1}} =$$

$$=\int_1^3 dx\int_{2x-1}^{5}(-2x+y+1)\,dy=\int_1^3\left((-2x)\cdot y\Big|_{2x-1}^{5}+\frac{1}{2}y^2\Big|_{2x-1}^{5}+y\Big|_{2x-1}^{5}\right)dx =$$

$$=\int_1^3\left((-2x)(5-(2x-1))+\frac{1}{2}\left(5^2-(2x-1)^2\right)+5-(2x-1)\right)dx =$$

$$=\int_1^3\left((-2x)(6-2x)+\frac{1}{2}\left(25-(4x^2-4x+1)\right)+6-2x\right)dx =$$

$$=\int_1^3\Big(\underline{\underline{-12x}}+\boxed{4x^2}+\overbrace{\frac{1}{2}(-4x^2+4x+24)}^{\boxed{-2x^2}+\underline{\underline{2x}}+12}+6\underline{\underline{-2x}}\Big)dx=\int_1^3(2x^2-12x+18)\,dx$$

$$=\frac{2}{3}x^3\Big|_1^3-\frac{12}{2}x^2\Big|_1^3+18x\Big|_1^3=\frac{2}{3}\cdot 26-\overset{48}{6\cdot 8}+\overset{36}{18\cdot 2}=\frac{52}{3}-12=\frac{16}{3}$$

Zapremina tijela ograničenog spomenutim ravnima iznosi $\frac{16}{3}$.

Izračunati zapreminu tijela ograničenog dijelom površi $(x^2+y^2+z^2)^3=\frac{a^6z^2}{x^2+y^2}$, $a>0$ u I oktantu.

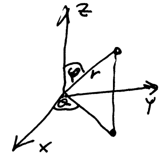
Rj.

Zapremina tijela ograničenog sa oblasti Ω se računa po formuli $V=\iiint\limits_{\Omega} dx\,dy\,dz$.

Datu površ $(x^2+y^2+z^2)^3=\frac{a^6z^2}{x^2+y^2}$ ne možemo skicirati.

Uvedimo sferne koordinate

$$x=r\sin\varphi\cos\theta$$
$$y=r\sin\varphi\sin\theta$$
$$z=r\cos\varphi$$



$$dx\,dy\,dz=r^2\sin\varphi\,dr\,d\varphi\,d\theta$$

$$\Omega\xrightarrow{\text{transformacija}}\Omega'$$

pa pokušajmo naći granice na osnovu date formule.

$$x^2+y^2+z^2=\underline{r^2\sin^2\varphi}\cos^2\theta+\underline{r^2\sin^2\varphi}\sin^2\theta+r^2\cos^2\varphi=r^2\sin^2\varphi+r^2\cos^2\varphi=r^2$$

$$(x^2+y^2+z^2)^3=(r^2)^3=r^6$$

$$z^2=r^2\cos^2\varphi$$

$$x^2+y^2=r^2\sin^2\varphi$$

$$(x^2+y^2+z^2)^3=\frac{a^6z^2}{x^2+y^2}$$

sad postaje $r^6=\frac{a^6r^2\cos^2\varphi}{r^2\sin^2\varphi}$

tj.

$$r^6=a^6\,ctg^2\varphi$$
$$r=\sqrt[6]{a^6\,ctg^2\varphi}$$
$$r=a\sqrt[3]{ctg\varphi}.$$

Na osnovu ove formule i znanja da je tijelo u I oktantu možemo zaključiti da je

$$\Omega':\begin{cases}0\le\varphi\le\frac{\pi}{2}\\ 0\le r\le a\sqrt[3]{ctg\varphi}\\ 0\le\theta\le\frac{\pi}{2}\end{cases}$$

$$V=\iiint\limits_{\Omega'} r^2\sin\varphi\,dr\,d\varphi\,d\theta=\int_0^{\pi/2}d\theta\int_0^{\pi/2}\sin\varphi\,d\varphi\int_0^{a\sqrt[3]{ctg\varphi}}r^2\,dr=\int_0^{\pi/2}d\theta\int_0^{\pi/2}\sin\varphi\frac{r^3}{3}\Big|_0^{a\sqrt[3]{ctg\varphi}}d\varphi$$

$$=\int_0^{\pi/2}d\theta\int_0^{\pi/2}\frac{a^3}{3}\sin\varphi\cdot\frac{\cos\varphi}{\sin\varphi}\,d\varphi=\frac{a^3}{3}\int_0^{\pi/2}d\theta\int_0^{\pi/2}\cos\varphi\,d\varphi=\frac{a^3}{3}\cdot\theta\Big|_0^{\frac{\pi}{2}}\cdot\sin\varphi\Big|_0^{\frac{\pi}{2}}=\frac{a^3\pi}{6}$$ tražena zapr.

Računanje težišta tijela

U slijedećim zadacima izračunajte koordinate težišta tijela (oblasti) Ω ograničenog datim površima!

1. $\Omega : z^2 = xy \wedge x = 5 \wedge y = 5 \wedge z = 0$.

Rješenje: najprije ćemo izračunati zapreminu date oblasti Ω. Očito je $0 \le z \le \sqrt{xy}$, a iz $z^2 = xy$ slijedi $xy \ge 0$, pa je $0 \le x \le 5 \wedge 0 \le y \le 5$. Zato je

$$V = \int_0^5 dx \int_0^5 dy \int_0^{\sqrt{xy}} dz = \int_0^5 \sqrt{x}dx \int_0^5 \sqrt{y}dy = \left(\int_0^5 \sqrt{x}dx\right)^2 = \frac{500}{9}.$$

Dalje imamo da je

$$\overline{x} = \frac{9}{500}\int_0^5 x\,dx \int_0^5 dy \int_0^{\sqrt{xy}} dz = \frac{9}{500}\int_0^5 x\sqrt{x}\,dx \int_0^5 \sqrt{y}dy = \frac{9}{500}\int_0^5 x^{\frac{3}{2}}\,dx \int_0^5 y^{\frac{1}{2}}dy = ... = 3.$$

Očigledno je $\overline{x} = \overline{y}$. Najzad,

$$\overline{z} = \frac{9}{500}\int_0^5 dx \int_0^5 dy \int_0^{\sqrt{xy}} z\,dz = \frac{9}{500}\cdot\frac{1}{2}\int_0^5 xdx \int_0^5 ydy = \frac{9}{1000}\left[\left.\frac{x^2}{2}\right|_0^5\right]^2 = \frac{9}{1000}\cdot\frac{25}{2}\cdot\frac{25}{2} = \frac{45}{32}.$$

Dakle, težište ima koordinate $T\left(3,3,\frac{45}{32}\right)$.

2. $\Omega : z = 3 - x^2 - y^2, z = 0.$

Rješenje: Uvešćemo cilindrične koordinate. Tada se Ω preslikava u oblast: $\Omega' : z = 3 - \rho^2, z = 0.$

U presjeku ove dvije površi se dobija kružnica $\rho^2 = 3 \Rightarrow \rho = \sqrt{3}.$ Zato je $0 \le \varphi \le 2\pi,\ 0 \le \rho \le \sqrt{3},\ 0 \le z \le 3 - \rho^2.$ Odatle slijedi:

$$V = \iiint_{\Omega'} \rho d\varphi\, d\rho\, dz = \int_0^{2\pi} d\varphi \int_0^{\sqrt{3}} \rho d\rho \int_0^{3-\rho^2} dz = 2\pi \int_0^{\sqrt{3}} \rho\left(3 - \rho^2\right) d\rho = 2\pi \int_0^{\sqrt{3}} \left(3\rho - \rho^3\right) d\rho =$$

$$= 2\pi\left.\left(3\frac{\rho^2}{2} - \frac{\rho^4}{4}\right)\right|_0^{\sqrt{3}} = 2\pi\left(\frac{9}{2} - \frac{9}{4}\right) = \frac{9\pi}{2}.$$

Sada možemo izračunati koordinate težišta tijela:

$$\overline{x} = \frac{1}{V}\iiint_{\Omega} x\,dx\,dy\,dz = \frac{2}{9\pi}\iiint_{\Omega'} \rho\cos\varphi \cdot \rho\, d\varphi\, d\rho\, dz = \frac{2}{9\pi}\int_0^{2\pi} \cos\varphi\, d\varphi \int_0^{\sqrt{3}} \rho^2 d\rho \int_0^{3-\rho^2} dz = 0,$$

jer je $\int_0^{2\pi} \cos\varphi\, d\varphi = 0.$ Na isti način dobijamo da je $\overline{y} = 0.$ I najzad,

$$\overline{z} = \frac{1}{V}\iiint_{\Omega'} \rho z\, d\varphi\, d\rho\, dz = \frac{2}{9\pi}\int_0^{2\pi} d\varphi \int_0^{\sqrt{3}} \rho d\rho \int_0^{3-\rho^2} z\,dz = \frac{4\pi}{9\pi}\int_0^{\sqrt{3}} \rho\frac{\left(3-\rho^2\right)^2}{2} d\rho.$$

U posljednjem integralu zgodno je uzeti smjenu $3 - \rho^2 = t.$ Dobija se dalje da je

$$\overline{z} = \frac{4}{9}\int_3^0 \frac{t^2}{2}\cdot\left(\frac{-1}{2}\right)dt = ... = 1.$$ Znači, $T\left(0,0,1\right)$.

Napomena: U nekim slučajevima možemo i bez računanja odmah zaključiti da je neka od koordinata težišta jednaka nuli. Radi se o slučajevima kada su jednačine površi koje opisuju oblast Ω simetrične u odnosu na neku od promjenljivih x, y ili z. Tako npr. u posljednjem zadatku, ako označimo $f\left(x,y,z\right) = z - \left(3 - x^2 - y^2\right) = x^2 + y^2 - z - 3$, imamo da je $f\left(x,y,z\right) = f\left(-x,y,z\right)$ i $f\left(x,y,z\right) = f\left(x,-y,z\right)$, što znači da je funkcija $f\left(x,y,z\right)$ simetrična u odnosu na x i u odnosu na y. Zato smo dobili da je $\overline{x} = \overline{y} = 0.$

Zadaci za samostalan rad:

3. $\Omega : z = \frac{y^2}{2},\ x = 0,\ y = 0,\ z = 0,\ 2x + 3y - 12 = 0.$
4. $x^2 + y^2 + z^2 = a^2,\ x^2 + y^2 = ax.$

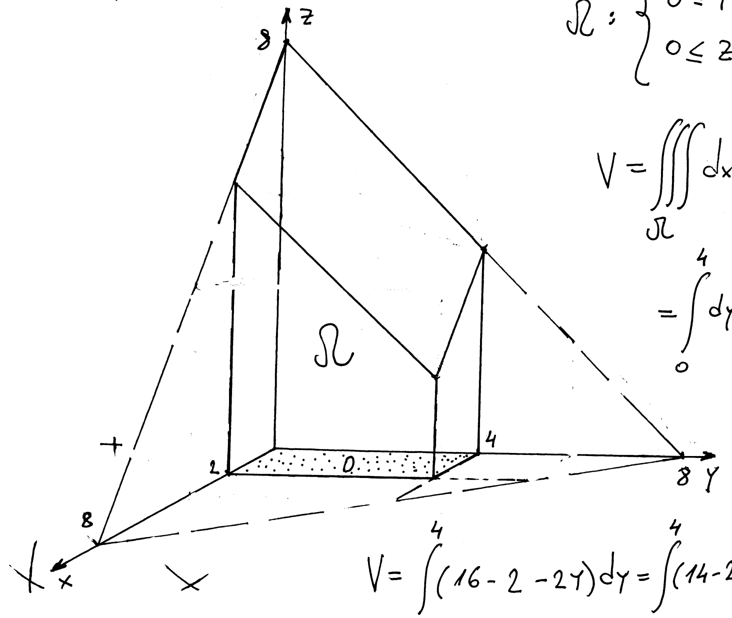
(#) Naći težište homogenog tijela ograničenog sa ravnima $x=0$, $y=0$, $z=0$, $x=2$, $y=4$ i $x+y+z=8$ (koso zasječen paralelopiped).

Rj.: Težište $T(x_T, y_T, z_T)$ homogenog tijela ograničenog sa oblašću Ω tražimo po formulama

$$x_T=\frac{1}{V}\iiint\limits_{\Omega} x\,dx\,dy\,dz,\quad y_T=\frac{1}{V}\iiint\limits_{\Omega} y\,dx\,dy\,dz,\quad z_T=\frac{1}{V}\iiint\limits_{\Omega} z\,dx\,dy\,dz$$

gdje je V zapremina tijela Ω.

Skicirajmo dato tijelo



$$\Omega:\begin{cases}0\le x\le 2\\ 0\le y\le 4\\ 0\le z\le 8-x-y\end{cases}$$

$$V=\iiint\limits_{\Omega} dx\,dy\,dz=\iint\limits_{D}(8-x-y)\,dx\,dy$$

$$=\int_0^4 dy\int_0^2(8-x-y)\,dx=$$

$$=\int_0^4\left(8x\Big|_0^2-\frac{1}{2}x^2\Big|_0^2-yx\Big|_0^2\right)dy$$

$$V=\int_0^4(16-2-2y)\,dy=\int_0^4(14-2y)\,dy=14y\Big|_0^4-2\cdot\frac{1}{2}y^2\Big|_0^4$$

$$V=14\cdot 4-16=4(14-4)=40$$

$$V=40$$

$$\iiint\limits_{\Omega} x\,dx\,dy\,dz=\int_0^4 dy\int_0^2 x\,dx\int_0^{8-x-y}dz=\int_0^4 dy\int_0^2(8x-x^2-yx)\,dx=\int_0^4\left(4x^2\Big|_0^2-\frac{1}{3}x^3\Big|_0^2-y\frac{1}{2}x^2\Big|_0^2\right)dy$$

$$=\int_0^4\left(16-\frac{8}{3}-2y\right)dy=\int_0^4\left(\frac{40}{3}-2y\right)dy=\frac{40}{3}y\Big|_0^4-2\cdot\frac{1}{2}y^2\Big|_0^4=\frac{160}{3}-16=\frac{112}{3}$$

$$\iiint\limits_{\Omega} y\,dx\,dy\,dz=\int_0^2 dx\int_0^4 y\,dy\int_0^{8-x-y}dz=\int_0^2 dx\int_0^4 y(8-x-y)\,dy=\int_0^2 dx\int_0^4(8y-xy-y^2)\,dy=$$

$$=\int_0^2\left(8\frac{1}{2}y^2\Big|_0^4-x\frac{1}{2}y^2\Big|_0^4-\frac{1}{3}y^3\Big|_0^4\right)dx=\int_0^2\left(64-8x-\frac{64}{3}\right)dx=\int_0^2\left(\frac{128}{3}-8x\right)dx=$$

$$=\frac{128}{3}x\Big|_0^2-8\cdot\frac{1}{2}x^2\Big|_0^2=\frac{256}{3}-16=\frac{208}{3}$$

$$\iiint\limits_{\Omega} z\,dx\,dy\,dz=\ \overset{\text{završiti za vježbu}}{\dots}\ =\frac{320}{3}$$

Prema tome, $x_T=\dfrac{1}{V}\iiint\limits_{\Omega} x\,dx\,dy\,dz=\dfrac{1}{\underset{\underset{\underset{5}{\cancel{10}}}{\cancel{20}}}{\cancel{40}}}\cdot\dfrac{\overset{\overset{\overset{14}{\cancel{28}}}{\cancel{56}}}{\cancel{112}}}{3}=\dfrac{14}{15}$

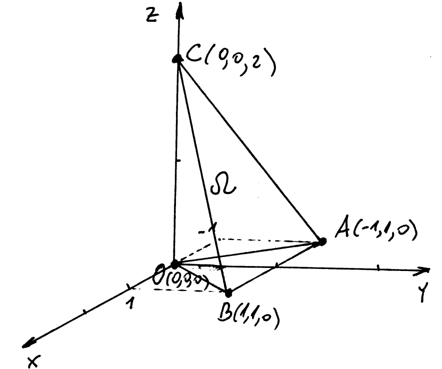
$$y_T=\frac{1}{V}\iiint\limits_{\Omega} y\,dx\,dy\,dz=\frac{1}{\underset{5}{\cancel{40}}}\cdot\frac{\overset{25}{\cancel{208}}}{3}=\frac{25}{15}$$

$$z_T=\frac{1}{V}\iiint\limits_{\Omega} z\,dx\,dy\,dz=\frac{1}{\underset{1}{\cancel{40}}}\cdot\frac{\overset{8}{\cancel{320}}}{3}=\frac{8}{3}$$

Težište homogenog tijela je $T\left(\frac{14}{15},\frac{25}{15},\frac{8}{3}\right)$.

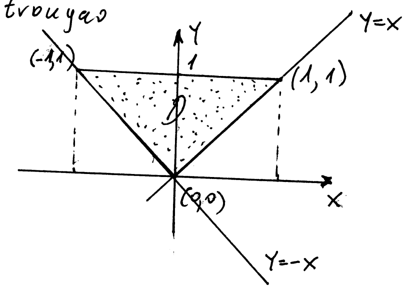
(#) Izračunati pomoću trostrukog integrala zapreminu i težište tetraedra OABC, ako je $O(0,0,0)$, $A(-1,1,0)$, $B(1,1,0)$, $C(0,0,2)$.

Rj: Skicirajmo dato tijelo



$$V=\iiint\limits_{\Omega} dx\,dy\,dz$$

Primjetimo da je projekcija tetraedra na xOy ravan trougao



Odredimo jednačinu ravni kroz tačke A, B i C

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$$

jednačina ravni kroz tri tačke

$$\begin{vmatrix} x-(-1) & y-1 & z-0 \\ 1-(-1) & 1-1 & 0-0 \\ 0-(-1) & 0-1 & 2-0 \end{vmatrix} = \begin{vmatrix} x+2 & y-1 & z \\ 2 & 0 & 0 \\ 1 & -1 & 2 \end{vmatrix} = (x+2)\cdot 0-(y-1)\cdot(4-0)+z\cdot(-2-0)$$

$$= -4y+4-2z$$

$$-4y+4-2z=0 \quad /:2$$

$$-2y-z+2=0$$

jednačina ravni koja prolazi kroz tačke A, B i C

$$V=\iiint\limits_{\Omega} dx\,dy\,dz=\int_0^1 dy\int_{-y}^{y} dx\int_0^{-2y+2} dz = \int_0^1 dy\int_{-y}^{y} z\Big|_0^{-2y+2} dx =$$

$$= \int_0^1 dy\int_{-y}^{y}(-2y+2)\,dx = \int_0^1\left(\underset{y-(-y)}{-2yx\Big|_{-y}^{y}}+2x\Big|_{-y}^{y}\right)dy = \int_0^1(-4y^2+4y)\,dy = -4\cdot\frac{1}{3}y^3\Big|_0^1+4\cdot\frac{1}{2}y^2\Big|_0^1 =$$

$$= -\frac{4}{3}+2=\frac{2}{3}$$ traženo rješenje

Zadaci za vježbu

Zapremina tela. II

U zadacima 3609 — 3625 pomoću trojnih integrala izračunati zapremine tela ograničenih datim površinama (parametre koji ulaze u uslove zadatka smatrati pozitivnim veličinama).

3609. Cilindrima $z=4-y^2$ i $z=y^2+2$ i ravnima $x=-1$ i $x=2$.

3610. Paraboloidima $z=x^2+y^2$ i $z=x^2+2y^2$ i ravnima $y=x$, $y=2x$ i $x=1$.

3611. Paraboloidima $z=x^2+y^2$ i $z=2x^2+2y^2$, cilindrom $y=x^2$ i ravni $y=x$.

3612. Cilindrima $z=\ln(x+2)$ i $z=\ln(6-x)$ i ravnima $x=0$, $x+y=2$ i $x-y=2$.

3613*. Paraboloidom $(x-1)^2+y^2=z$ i ravni $2x+z=2$.

3614*. Paraboloidom $z=x^2+y^2$ i ravni $z=x+y$.

3615*. Sferom $x^2+y^2+z^2=4$ i paraboloidom $x^2+y^2=3z$.

3616. Sferom $x^2+y^2+z^2=R^2$ i paraboloidom $x^2+y^2=R(R-2z)$ $(z\geqslant 0)$.

3617. Paraboloidom $z=x^2+y^2$ i konusom $z^2=xy$.

3618. Sferom $x^2+y^2+z^2=4Rz-3R^2$ i konusom $z^2=4(x^2+y^2)$ (misli se na deo loptine zapremine koji leži unutar konusa).

3619*. $(x^2+y^2+z^2)^2=a^3x$.

3620. $(x^2+y^2+z^2)^2=axyz$.

3621. $(x^2+y^2+z^2)^3=a^2z^4$. **3622.** $(x^2+y^2+z^2)^3=\dfrac{a^6z^2}{x^2+y^2}$,

3623. $(x^2+y^2+z^2)^3=a^2(x^2+y^2)^2$.

3624. $(x^2+y^2)^2+z^4=a^3z$.

3625. $x^2+y^2+z^2=1$, $x^2+y^2+z^2=16$, $z^2=x^2+y^2$, $x=0$, $y=0$, $z=0$ $(x>0,\ y>0,\ z\geqslant 0)$.

Težišta homogenih tela

U zadacima 3666 — 3672 naći težišta homogenih tela ograničenih datim površinama.

3666. Ravnima $x=0$, $y=0$, $z=0$, $x=2$, $y=4$ i $x+y+z=8$ (koso zasečeni paralelepiped).

3667. Elipsoidom $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ i koordinatnim ravnima (misli se na deo elipsoida koji leži u prvom oktantu).

3668. Cilindrom $z=\dfrac{y^2}{2}$ i ravnima $x=0$, $y=0$, $z=0$ i $2x+3y-12=0$.

3669. Cilindrima $y=\sqrt{x}$, $y=2\sqrt{x}$ i ravnima $z=0$ i $x+z=0$.

3670. Paraboloidom $z=\dfrac{x^2+y^2}{2a}$ i sferom $x^2+y^2+z^2=3a^2$ $(z>0)$.

3671. Sferom $x^2+y^2+z^2=R^2$ i konusom $z\,\mathrm{tg}\,\alpha=\sqrt{x^2+y^2}$ (loptin isečak).

3672. $(x^2+y^2+z^2)^2=a^3z$.

Rješenja

3666. $\xi=\frac{14}{15}$, $\eta=\frac{26}{15}$, $\zeta=\frac{8}{3}$. **3667.** $\xi=\frac{3}{8}a$, $\eta=\frac{3}{8}b$, $\zeta=\frac{3}{8}c$.

3668. $\xi=\frac{6}{5}$, $\eta=\frac{12}{5}$, $\zeta=\frac{8}{5}$. **3669.** $\xi=\frac{18}{7}$, $\eta=\frac{15}{16}\sqrt{6}$, $\zeta=\frac{12}{7}$.

3670. $\xi=0$, $\eta=0$, $\zeta=\frac{5a}{83}(6\sqrt{3}+5)$.

3671. $\xi=0$, $\eta=0$, $\zeta=\frac{3R}{8}(1+\cos\alpha)$. **3672.** $\xi=0$, $\eta=0$, $\zeta=\frac{9a}{20}$.

Rješenja

3609. 8.

3610. $\frac{7}{12}$. **3611.** $\frac{3}{35}$.

3612. $4(4-3\ln 3)$.

3613*. $\frac{\pi}{2}$. Projekcija tela na ravan xOy je krug.

3614. $\frac{\pi}{8}$. Preneti koordinatni početak u tačku $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$.

3615*. $\frac{19}{6}\pi$ i $\frac{15}{2}\pi$. Preći na cilindrične koordinate.

3616. $\frac{5}{12}\pi R^3$. **3617.** $\frac{\pi}{96}$.

3618. $\frac{92}{75}\pi R^2$.

3619*. $\frac{1}{3}\pi a^3$. Preći na sferne koordinate.

3620. $\frac{a^3}{360}$. **3621.** $\frac{4}{21}\pi a^3$.

3622. $\frac{4}{3}\pi a^3$. **3623.** $\frac{64}{105}\pi a^3$.

3624. $\frac{\pi^2a^3}{6}$. **3625.** $\frac{21(2-\sqrt{2})}{4}\pi$.

Pismeni dio ispita iz Matematike II (MF), 02.02.2012.

Grupa A

1. Izračunati integral $A=\int\limits_{\frac{\pi}{2}}^{2\operatorname{arctg}3}\frac{\operatorname{tg}^2\frac{x}{2}}{2\sin x-\cos x+1}dx.$
2. Odrediti ekstreme funkcije $z=x+\frac{y^2}{2x}+\frac{x^2}{y}+\frac{5}{2x}.$
3. Dat je trostruki integral $\int\limits_0^{2\pi}d\varphi\int\limits_0^2\rho^3d\rho\int\limits_0^{\sqrt{4-\rho^2}}dz$ u cilindričnim koordinatama. Skicirati oblast integracije i izračunati taj integral prelazeći na sferne koordinate.
4. Izračunati površinski integral $I=\iint\limits_S ydydz-xdzdx+zdxdy,$ ako je S donja strana dijela površi $z=\sqrt{x^2+y^2}$ kojeg isjeca površ $x^2+y^2=y.$

Grupa B

1. Izračunati integral $B=\int\limits_{-2\operatorname{arctg}2}^{2\operatorname{arctg}3}\frac{\operatorname{tg}\frac{x}{2}}{2\sin x+6\cos x+7}dx.$
2. Odrediti ekstreme funkcije $z=\left(\frac{1}{9}x^2+y^2\right)e^{-\frac{x}{3}+y}.$
3. Izračunati trostruki integral $I=\iiint\limits_\Omega\sqrt{x^2+y^2+z^2}\,dx\,dydz,$ pri čemu je Ω unutrašnjost lopte $x^2+y^2+z^2=x.$
4. Izračunati površinski integral $I=\iint\limits_S x^2dydz+y^2dzdx+z^2dxdy,$ ako je S vanjska strana tijela određenog ravnima $x=0, y=0, z=h$ i dijelom konusa $x^2+y^2=z^2$ u prvom oktantu.

Stari program:

1. Ispitati konvergenciju reda $\sum\limits_{n=1}^{\infty}3^{-n}\left(\frac{n+1}{n}\right)^{n^2}.$
2. Riješiti diferencijalnu jednačinu $y^{iv}+y''=x^2\cos x.$
3. Izračunati trostruki integral $I=\iiint\limits_\Omega\sqrt{x^2+y^2+z^2}\,dx\,dydz,$ pri čemu je Ω unutrašnjost lopte $x^2+y^2+z^2=x.$
4. Izračunati površinski integral $I=\iint\limits_S x^2dydz+y^2dzdx+z^2dxdy,$ ako je S vanjska strana tijela određenog ravnima $x=0, y=0, z=h$ i dijelom konusa $x^2+y^2=z^2$ u prvom oktantu.

Pismeni dio ispita iz Matematike II (MF), 17.02.2012.

GRUPA A

1. Izračunati zapreminu tijela koje nastaje rotacijom oko x – ose figure određene parabolom $2y^2=ax\,(a>0)$ i pravama $y=0,\ x+y=a.$
2. Izračunati dvostruki integral $I=\iint\limits_D\frac{xy\sqrt{1-x^2-y^2}}{2x^2+y^2}dx,$ ako je D dio kruga $x^2+y^2\le1$ u prvom kvadrantu.
3. Izračunati pomoću krivolinijskog integrala površinu figure omeđene krivom $x=\frac{5at^2}{1+t^5}, y=\frac{5at^3}{1+t^5},\ 0\le t\le1.$
4. Izračunati integral $I(\alpha)=\int\limits_0^1\frac{\ln\left(1-\alpha^2x^2\right)}{x^2\sqrt{1-x^2}}dx\ \left(\alpha^2<1\right)$ pomoću diferenciranja po parametru $\alpha.$

$\sqrt{}$

GRUPA B

1. Izračunati zapreminu tijela koje nastaje rotacijom oko x – ose figure u drugom kvadrantu određene parabolom $y^2=-\frac{ax}{2}(a>0)$ i pravama $y=0,\ y-x=a.$
2. Izračunati dvostruki integral $I=\iint\limits_D\frac{1}{\left(x^2+y^2\right)\left(\sqrt[3]{x^2+y^2}+1\right)}dx,$ ako je oblast D određena nejednačinama $x^2-y^2\le0,\ 1\le x^2+y^2\le4.$
3. Izračunati pomoću krivolinijskog integrala površinu figure omeđene krivom $x=\frac{at}{\left(1+t\right)^4},\ y=\frac{at^2}{\left(1+t\right)^4},0\le t\le1.$
4. Izračunati integral $I(\alpha)=\int\limits_0^1\frac{\operatorname{arctg}(\alpha x)}{x\sqrt{1-x^2}}dx\left(\alpha>0\right)$ pomoću diferenciranja po parametru $\alpha.$

Stari program:

1. Naći oblast konvergencije reda: $\sum\limits_{n=1}^{\infty}\frac{\left(n+1\right)^2\left(x+1\right)^n}{\left(3n+1\right)^3\cdot4^{2n-2}}.$
2. Riješiti diferencijalnu jednačinu $y'-\frac{xy}{2\left(x^2-1\right)}-\frac{x}{2y}=0.$
3. Izračunati pomoću krivolinijskog integrala površinu figure omeđene krivom $x=\frac{at}{\left(1+t\right)^4},\ y=\frac{at^2}{\left(1+t\right)^4},0\le t\le1.$
4. Izračunati integral $I(\alpha)=\int\limits_0^1\frac{\operatorname{arctg}(\alpha x)}{x\sqrt{1-x^2}}dx\left(\alpha>0\right).$ pomoću diferenciranja po parametru $\alpha.$

Drugi parcijalni ispit, 08.06.2012.

GRUPA A

1. Izračunati zapreminu tijela u oblasti $\Omega: x^2+y^2+z^2 \le 36, x^2+y^2 \ge z^2, z \le 0.$
2. Izračunati krivolinijski integral $\int\limits_c \sqrt{2y}\,ds,$ ako je c kriva $x=t, y=\frac{1}{2}t^2, z=\frac{1}{3}t^3, 0 \le t \le 1.$
3. Izračunati površinski integral $I=\iint\limits_S 3zdS, S: z=2-x^2-y^2, z \ge 0.$

Drugi parcijalni ispit, 08.06.2012.

GRUPA B

1. Izračunati zapreminu tijela u oblasti $\Omega: x^2+y^2+z^2 \ge 4,\ x^2+y^2 \le 3z,\ z \le \sqrt{x^2+y^2}.$
2. Izračunati krivolinijski integral $\int\limits_c (x+z)ds,$ ako je c kriva $x=t,\ y=\frac{3}{\sqrt{2}}t^2,\ z=t^3, 0 \le t \le 1.$
3. Izračunati površinski integral $E=\iint\limits_S \left(\sqrt{1-z^2}-z\right)dS,\ \ S: z^2=x^2+y^2,\ 0 \le z \le 1.$

Pismeni dio ispita iz Matematike II, 21.06.2012.

GRUPA A

1. Izračunati površinu figure koja je određena parabolom $y^2=2ax,\ a>0$ i normalom na parabolu koja zaklapa ugao od 135^0 sa x – osom.
2. Naći uslovne ekstreme funkcije $z=ax+by,$ ako je $x^2+y^2=1.$
3. Izračunati trostruki integral $I=\iiint\limits_\Omega \frac{x+y}{a^2+z^2}dxdydz,$ ako je Ω oblast ograničena ravnima $x=0, y=0, x+y+z=a, x+y-z=a,\ a>0.$
4. Izračunati površinski integral druge vrste $I=\iint\limits_S x^2dydz+y^2dzdx+z^2dxdy,$ ako je S vanjski dio površi $z^2=2x^2+2y^2,\ 0 \le z \le 4.$

GRUPA B

1. Izračunati površinu figure koju u ravni određuju linije: $y=\frac{b^3}{b^2+x^2},\ 2by=x^2, b>0.$
2. Naći jednačinu tangentne ravni na površ $z=2cxy,$ koja prolazi kroz tačku $A(1,0,-4c)$ i okomita je na ravan $x=y.$
3. Izračunati trostruki integral $I=\iiint\limits_\Omega \frac{y+z}{a^2+x^2}dxdydz,$ ako je Ω oblast ograničena ravnima $y=0, z=0, x+y+z=a, y+z-x=a,\ a>0.$
4. Izračunati površinski integral druge vrste $I=\iint\limits_S (2xz+z\sin 2x+x+y)dydz+(4yz\sin^2 x+y+z)dzdx+(x+y-2z^2)dxdy,$ ako je S vanjski dio oblasti određene površima $z=4-2x^2-y^2,\ z=-x^2.$

Stari program

1. Naći sumu reda $\sum\limits_{n=1}^{\infty}\frac{1}{(3n-1)(3n+2)(3n+5)}.$
2. Riješiti diferencijalnu jednačinu $x^2(x-1)y'-y^2-x(x-2)y=0.$
3. Izračunati trostruki integral $I=\iiint\limits_\Omega \frac{y+z}{a^2+x^2}dxdydz,$ ako je Ω oblast ograničena ravnima $y=0, z=0, x+y+z=a, y+z-x=a,\ a>0.$
4. Izračunati površinski integral druge vrste $I=\iint\limits_S (2xz+z\sin 2x+x+y)dydz+(4yz\sin^2 x+y+z)dzdx+(x+y-2z^2)dxdy,$ ako je S vanjski dio oblasti određene površima $z=4-2x^2-y^2,\ z=-x^2.$

Pismeni ispit iz Matematike II, 06.07.2012.

GRUPA A

1. Izračunati integral $\int\limits_0^2 x\sqrt{4+x^2}\,\text{arctg}\frac{x}{2}dx.$
2. Promijeniti poredak integracije u integralu $I=\int\limits_{-7}^{-1}dy\int\limits_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}}f(x,y)dx.$
3. Odrediti brojeve a i b tako da vektorsko polje $\vec{v}=\left(yz+axy,xz+bx^2+yz^2,axy+y^2z\right)$ bude potencijalno i za dobijeno polje izračunati njegovu cirkulaciju duž pravolinijske konture od tačke $A(1,1,1)$ prema tački $B(2,2,2)$.
4. Izračunati krivolinijski integral prve vrste $\int\limits_c (x+y)ds,$ ako je c desna latica lemniskate

   $\rho=a\sqrt{\cos 2\varphi}.$

GRUPA B

1. Izračunati integral $\int\limits_0^3 \arcsin\sqrt{\frac{x}{x+1}}dx.$
2. Promijeniti poredak integracije u integralu $\int\limits_0^1 dy\int\limits_{\frac{y^2}{2}}^{\sqrt{3-y^2}}f(x,y)dx.$
3. Dokazati da je vektorsko polje $\vec{v}=\left(2xz,2yz,x^2+y^2-z^2\right)$ potencijalno i naći tok (fluks) tog polja kroz vanjsku stranu sfere $x^2+y^2+(z-1)^2=1$.
4. Izračunati krivolinijski integral prve vrste $\int\limits_c (x-y+2z)ds,$ ako je c kontura trougla

   $ABC,\ A(0,0,0),B(14,0,0),C\left(9,\frac{36}{5},\frac{48}{5}\right).$

Stari program

1. Razviti u Fourierov red funkciju $f(x)=\frac{\pi-x}{2},x\in[0,2\pi].$
2. Riješiti diferencijalnu jednačinu $y'''-4y=x.$
3. Odrediti brojeve a i b tako da vektorsko polje $\vec{v}=\left(yz+axy,xz+bx^2+yz^2,axy+y^2z\right)$ bude potencijalno i za dobijeno polje izračunati njegovu cirkulaciju duž pravolinijske konture od tačke $A(1,1,1)$ prema tački $B(2,2,2)$.
4. Izračunati krivolinijski integral prve vrste $\int\limits_c (x+y)ds,$ ako je c desna latica lemniskate

   $\rho=a\sqrt{\cos 2\varphi}.$

Pismeni ispit iz Matematike II, 06.09.2012.

GRUPA A

1. Odrediti zapreminu tijela nastalog rotacijom krive $(y-2)^2=x(4-x)$ oko $y-$ ose.
2. Izračunati trostruki integral $\iiint\limits_{\Omega}(x+y+z)\,dx\,dy\,dz,$ ako je

   $\Omega: x^2+y^2\le 2a\ ,\ x\overset{2}{z}+y^2+z^2\le 3a^2,\ a>0.$
3. Izračunati pomoću Greenove formule krivolinijski integral $\oint\limits_c\frac{xdy-\left(y+x^3\right)dx}{\left(x^2+y^2+2y\right)^3},$ ako je c

   pozitivno orjentisana kontura kružnice $x^2+y^2+2y=1.$
4. Izračunati površinski integral $\iint\limits_S zdydz+xdzdx+ydxdy,$ ako je S dio sfere $x^2+y^2+z^2=a^2$

   unutar cilindra $x^2+y^2=ax,\ a>0.$

GRUPA B

1. Odrediti zapreminu tijela nastalog rotacijom krive $(x+1)^2=-y(y+2)$ oko $x-$ ose.
2. Izračunati trostruki integral $\iiint\limits_{\Omega}\left(4x^2+y^3-1\right)dx\,dy\,dz,$ ako je oblast Ω ograničena površima

   $z=x^2-2x+2y^2+4y-2$ i $z=4x^2-2x+5y^2+4y-14.$
3. Izračunati pomoću Greenove formule krivolinijski integral

   $\oint\limits_c\left(x^2y+\frac{y^3}{3}+ye^{xy}\right)dx+\left(x+xe^{xy}\right)dy,$ ako je c pozitivno orjentisana kontura određena

   linijama $y=\sqrt{1-x^2},y=0.$
4. Izračunati površinski integral $\iint\limits_S xzdydz+xydzdx+yzdxdy,$ ako je S vanjska strana omotača

   tijela koje pripada prvom oktantu i ograničeno je cilindrom $x^2+y^2=1,$ te ravnima
   $x=0,y=0,z=0,z=2.$

Stari program:

1. Razviti u Fourierov red funkciju $f(x)=\frac{\pi-x}{2},x\in[0,2\pi].$
2. Riješiti diferencijalnu jednačinu $y'''-4y=x.$
3. Izračunati pomoću Greenove formule krivolinijski integral $\oint\limits_c\frac{xdy-\left(y+x^3\right)dx}{\left(x^2+y^2+2y\right)^3},$ ako je c

   pozitivno orjentisana kontura kružnice $x^2+y^2+2y=1.$
4. Izračunati površinski integral $\iint\limits_S zdydz+xdzdx+ydxdy,$ ako je S dio sfere $x^2+y^2+z^2=a^2$

   unutar cilindra $x^2+y^2=ax,\ a>0.$

Pismeni dio ispita **iz Matematike II**, 03.02.2011.

GRUPA A

1. Naći površinu figure koja je ograničena linijama $y=-x^2, x-y-2=0$.
2. Naći ekstreme funkcije $z=x^3+3xy^2-15x-12y$.
3. Naći zapreminu tijela ograničenog ravnima $x=1, x=3, y=1, y=5,\ 2x-y+z-1=0,\ z=0$.
4. Izračunati krivolinijski integral $I=\int\limits_c z\sqrt{x^2+y^2+2z^2}\,ds,$ ako je c kriva

$$x=\frac{r\sqrt{2}}{2}\cos t, y=\frac{r\sqrt{2}}{2}\cos t, z=r\sin t, t\in[0,\pi].$$

GRUPA B

1. Izračunati površinu rotacionog tijela koje se dobije rotacijom parabole $y^2=4x$ od tačke $x=0$ do tačke $x=2$.
2. Naći uslovne ekstreme funkcije $z=6+4x+3y$ uz uslov $x^2+y^2=1$.
3. Naći zapreminu tijela ograničenog ravnima $x=-1,\ x=2,\ y=-2,\ y=2,\ 4x-3y+z-2=0,\ z=0$.
4. Izračunati krivolinijski integral $I=\int\limits_c y\sqrt{x^2+4y^2+z^2}\,ds,$ ako je c kriva

$$x=\frac{a\sqrt{6}}{3}\sin t, y=a\cos t, z=\frac{a\sqrt{3}}{3}\sin t, t\in\left[0,\frac{\pi}{2}\right].$$

Pismeni dio ispita iz **Matematike II,** 18.02.2011

GRUPA A

1. Izračunati integrale: $I_1=\int\limits_1^3 x^3\sqrt{x^2-1}\,dx, I_2=\int\limits_0^{\frac{\pi}{2}}\frac{dx}{1+\sin x+\cos x}.$
2. Izmjeniti poredak integracije u integralu $I=\int\limits_1^2 dx\int\limits_{2-x}^{\sqrt{2x-x^2}} f(x,y)\,dy$.
3. Izračunati površinski integral $P=\iint\limits_S (z^2+1)\,dS,$ S je dio sfere

   $x^2+y^2+z^2=4$ u prvom oktantu.
4. Izračunati integral $I(\alpha)=\int\limits_0^\infty \frac{1-e^{-\alpha x^2}}{xe^{x^2}}dx$ pomoću diferenciranja po parametru

   ako je $\alpha>-1$.

GRUPA B

1. Izračunati integrale: $I_1=\int\limits_0^4\frac{dx}{1+\sqrt{2x+1}},\ I_2=\int\limits_0^{\frac{\pi}{4}}\frac{\sin x}{\sin x+\cos x}dx.$
2. Izmjeniti poredak integracije u integralu $I=\int\limits_0^1 dx\int\limits_x^{\sqrt{2-x^2}} f(x,y)\,dy$.
3. Izračunati površinski integral $\iint\limits_{(S)}\sqrt{-x^2+4}\ dS$, gdje je (S) omotač površi

   $\frac{x^2}{4}+\frac{y^2}{4}=\frac{z^2}{9}, 0\le z\le 3.$
4. Izračunati pomoću diferenciranja po parametru integral

   $$I(\alpha)=\int\limits_0^{\frac{\pi}{2}}\ln\left(\sin^2 x+\alpha^2\cos^2 x\right)\,dx,\ \ \alpha>0.$$

Pismeni dio ispita iz **Matematike II**, 23.06.2011.

GRUPA A

1. Izračunati dužinu luka krive $y=\ln\frac{e^x-1}{e^x+1}$ od tačke sa apscisom $x=1$ do tačke sa apscisom $x=2$.
2. Izračunati pomoću dvostrukog integrala zapreminu tijela kojeg ograničavaju površi $x^2+y^2-2az=0, \left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right),\ a>0.$
3. Izračunati pomoću Greenove formule krivolinijski integral

   $I=\oint\limits_c\sqrt{x^2+y^2}\,dx+y\left[xy+\ln\left(x+\sqrt{x^2+y^2}\right)\right]dy,$ ako je c kontura koja ograničava oblast $y^2\le 2x-2,\ x\le 2,\ y\ge 0.$
4. Izračunati površinski integral $I=\iint\limits_S\left(x+y+z^2\right)dS,$ ako je S polulopta

   $x^2+y^2+z^2=9, z\ge 0.$

GRUPA B

1. Izračunati dužinu luka krive $y=a\ln\frac{a^2}{a^2-x^2}$ $(a>0)$ od tačke $A(0,0)$ do tačke

   $B\left(\frac{a}{2}, a\ln\frac{4}{3}\right).$
2. Izračunati pomoću dvostrukog integrala zapreminu tijela kojeg ograničavaju površi $x^2+y^2=4\ (x\ge 0),\ x^2-y^2=1\ (x\ge 1), z=4-x^2\ (z\ge 0)$ i ravan $z=0$.

3. Izračunati krivolinijski integral $\oint\limits_c x ds,$ ako je c lemniskata $\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right),\ a>0.$

4. Izračunati površinski integral $I=\iint\limits_S (x-y+z)dS,$ ako je S polulopta $x^2+y^2+z^2=4, z\geq 0.$

Pismeni dio ispita iz **Matematike II**, 08.07.2011.

Grupa A

1. Odrediti jednačinu tangentne ravni na površ $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, koja je normalna na pravoj $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$.

2. Izračunati integral po glatkom luku koji spaja tačke A i B

$$\int\limits_{\overset{\frown}{AB}}\left(1-\frac{1}{y}+\frac{y}{z}\right)dx+\left(\frac{x}{z}+\frac{x}{y^2}\right)dy-\frac{xy}{z^2}dz\ ,\ A(1,1,1), B(1,2,3), \overset{\frown}{AB}\subset\left\{(x,y,z): x>0, y>0, z>0\right\}$$

3. Izračunati zapreminu onog dijela lopte $x^2+y^2+z^2=1$ koji se nalazi unutar cilindra $x^2+(y-1)^2=1$.

4. Dato je vektorsko polje $\vec{A}=(e^x z-2xy, 1-x^2, e^x+z)$. Pokazati da je polje A potencijalno i odrediti mu potencijal. Izračunati integral $\int\limits_L \vec{A}\cdot d\vec{r}$, gdje je L duž PQ, P (0, 1, -1), Q (2, 3, 0), orijentisana od P prema Q.

Grupa B

1. Dokazati da proizvoljna tangentna ravan površi S: $xyz=a^3\,(a>0, \text{konstanta})$ obrazuje sa koordinatnim ravnima tetraedar stalne zapremine $\left(V=\frac{9}{2}a^3\right)$.

2. Izračunati integral po glatkom luku koji spaja tačke A i B

$$\int\limits_{\overset{\frown}{AB}}\frac{zxdy+xydz-yzdx}{(x-yz)^2}\ A(7,2,3),\ B(5,3,1), \left(z\neq\frac{x}{y}\right).$$

3. Izračunati zapreminu tijela koje je ograničeno površima $x^2+y^2=y, x^2+y^2=2y, z=y^2, z=0$.

4. Dato je vektorsko polje $\vec{A}=(2x(y^2+z^2)+yz, 2y(z^2+x^2)+xz, 2z(x^2+y^2)+xy)$. Pokazati da je polje A potencijalno i odrediti mu potencijal. Izračunati fluks vektorskog polja $\vec{A}$ kroz spoljnu stranu polusfere $\Gamma: x^2+y^2+z^2-2z=0, y\geq 0$

Pismeni dio ispita iz **Matematike II,** 15.09.2011.

GRUPA A

1. Izračunati površinu figure koju određuju prava $2x+3\sqrt{3}y-12=0$ i dio elipse $4x^2+9y^2=36$ u prvom kvadrantu.

2. Promijeniti poredak integracije i izračunati dvostruki integral $I=\int\limits_0^2 ydy\int\limits_{\frac{1}{2}}^{2-\frac{y^2}{8}}\frac{dx}{\sqrt{x^5}}.$

3. Izračunati površinu dijela površi $x^2+y^2+z-1=0$ koji se nalazi iznad ravni $z=0.$

4. Dati su krivolinijski integrali $I_1=\int\limits_{c_1}\frac{xdy-ydx}{x^2+y^2},\ I_2=\int\limits_{c_2}\frac{xdy-ydx}{x^2+y^2},$ gdje je c_1 duž $\overline{AB},\ A(1,2), B(-1,4),$ orijentisana od tačke A prema tački B, a c_2 je parabola koja prolazi kroz tačke $A(1,2), B(-1,4)$ i $C\left(\frac{-1}{2},\frac{11}{4}\right)$. Dokazati da je $I_1=I_2$ i izračunati taj broj.

GRUPA B

1. Izračunati površinu krivolinijskog četverougla omeđenog parabolama $y=x^2,\ y=\frac{x^2}{3},\ y^2=2x,\ y^2=3x.$

2. Promijeniti poredak integracije i izračunati dvostruki integral

$$I=\int\limits_0^a ydy\int\limits_0^{a-\sqrt{a^2-y^2}}\frac{x\ln(x+a)dx}{(x-a)^2}.$$

3. Izračunati površinu dijela sfere $x^2+y^2+z^2=a^2$ koji se nalazi u unutrašnjosti cilindra $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ 0<b<a.$

4. Izračunati krivolinijski integral $I=\oint\limits_c\frac{1}{x}\text{arctg}\frac{y}{x}dx+y^3e^{-y}dy,$ ako je c pozitivno orjentisana kontura oblasti određene isječkom kružnog prstena $1\leq x^2+y^2\leq 4,\ 0\leq y\leq x.$

Pismeni dio ispita iz **Matematike II,** 23.09.2011.

GRUPA A

1. Izračunati površinu figure koju određuju prava $2x+3\sqrt{3}y-12=0$ i dio elipse $4x^2+9y^2=36$ u prvom kvadrantu.

2. Promijeniti poredak integracije i izračunati dvostruki integral $I=\int\limits_0^2 ydy\int\limits_{\frac{1}{2}}^{2-\frac{y^2}{8}}\frac{dx}{\sqrt{x^5}}.$

3. Izračunati površinu dijela površi $x^2+y^2+z-1=0$ koji se nalazi iznad ravni $z=0.$

4. Dati su krivolinijski integrali $I_1=\int\limits_{c_1}\frac{xdy-ydx}{x^2+y^2}$, $I_2=\int\limits_{c_2}\frac{xdy-ydx}{x^2+y^2}$, gdje je c_1 duž $\overline{AB}$, $A(1,2), B(-1,4)$, orjentisana od tačke A prema tački B, a c_2 je parabola koja prolazi kroz tačke $A(1,2), B(-1,4)$ i $C\left(\frac{-1}{2},\frac{11}{4}\right)$. Dokazati da je $I_1=I_2$ i izračunati taj broj.

GRUPA B

1. Izračunati površinu krivolinijskog četverougla omeđenog parabolama
$y=x^2,\ y=\frac{x^2}{3},\ y^2=2x,\ y^2=3x.$
2. Promijeniti poredak integracije i izračunati dvostruki integral
$$I=\int\limits_0^a ydy\int\limits_0^{a-\sqrt{a^2-y^2}}\frac{x\ln(x+a)dx}{(x-a)^2}.$$
3. Izračunati površinu dijela sfere $x^2+y^2+z^2=a^2$ koji se nalazi u unutrašnjosti cilindra $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ 0<b<a.$
4. Izračunati krivolinijski integral $I=\oint\limits_c\frac{1}{x}\operatorname{arctg}\frac{y}{x}dx+y^3e^{-y}dy,$ ako je c pozitivno orjentisana kontura oblasti određene isječkom kružnog prstena
$1\le x^2+y^2\le 4,\ 0\le y\le x.$

Pismeni dio ispita iz **Matematike II,** oktobar 2011.

1. Odrediti jednačinu tangentne ravni na površ $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, koja je normalna na pravoj $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$.
2. Promijeniti poredak integracije i izračunati dvostruki integral
$$I=\int\limits_0^a ydy\int\limits_0^{a-\sqrt{a^2-y^2}}\frac{x\ln(x+a)dx}{(x-a)^2}.$$
3. Izračunati krivolinijski integral $\oint\limits_c xds,$ ako je c lemniskata
$\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right),\ a>0.$
4. Dato je vektorsko polje $\vec{A}=(2x(y^2+z^2)+yz,2y(z^2+x^2)+xz,2z(x^2+y^2)+xy)$.
Pokazati da je polje A potencijalno i odrediti mu potencijal. Izračunati fluks vektorskog polja $\vec{A}$ kroz spoljnu stranu polusfere $\Gamma: x^2+y^2+z^2-2z=0, y\ge 0$.